# Mathematical and Physical Basis of Medical Biophysics

Lecture 1 & 2: Mathematical Tools

6th & 9th September 2016

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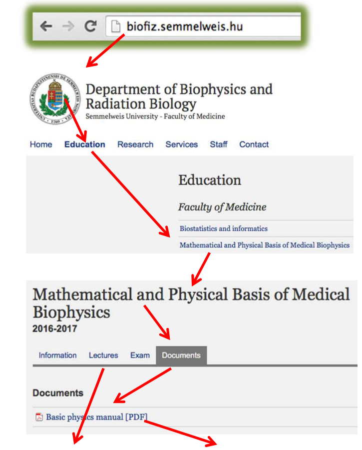
# How to Get Prepared?

- university = **autonomous learning**
- sources:
  - **your** notes made during lectures *(Tuesday $18^{10}$–$19^{30}$; Friday $17^{40}$–$19^{00}$; EOK "Szent-Györgyi Albert" lecture hall;* ***only for the first four weeks****)*
  - Tölgyesi: *Mathematical and Physical Basis of Medical Biophysics* (2016)
  - homepage: biofiz.semmelweis.hu
    - subject requirements
    - lecture schedule and slides
    - textbook



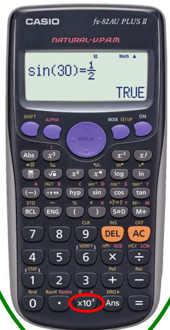

Lecture slides
(will be uploaded one by one)

Textbook



# How to Use Scientific Notation?


best calculator for a medical student





still okay



not allowed




# Use of Symbols in Science

In science we use a large number of **Latin** and **Greek** letters (and their combinations) as symbols, so it is inevitable to learn the Greek alphabet.

However, the number of quantities and units is much greater than the number of available letters, and this can lead to confusion. Your help: CONTEXT

**ambiguity!** ← **BEWARE!** → **inconsistency!**

**for example:**

**c and C**: specific heat capacity, a general constant, Celsius, carbon, capacitance and capacitor, coulomb, centi-, concentration (many different kinds), speed of sound, speed of light

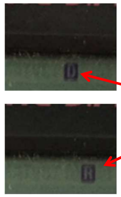
**for example:**

- density ← ***ρ*** [rho], ***d***
- speed ← ***v***, ***c***
- multiplication ← **×**, **·**, *****
- frequency ← ***f***, ***ν*** [nu]
- proportionality ← **∝**, **~**

# Angles

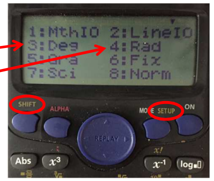
– shift
– setup
– 3 (for degrees)
– 4 (for radians)

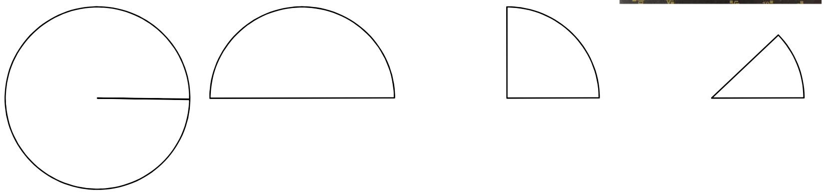
D: degrees mode
R: radians mode



**revolution**: one turn
**degree**: practical, traditional unit
**radian**: scientific unit, arc/radius

1 revolution = 360° = 2π rad



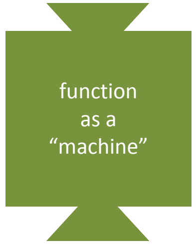
| one **revolution** | half **revolution** | quarter **revolution** | 1/8 **revolution** |
|---|---|---|---|
| 360° **degrees** | 180° **degrees** | 90° **degrees** | 45° **degrees** |
| 2π **radian** | π **radian** | π/2 **radian** | π/4 **radian** |



# What is a Function?

Unambiguous assignment of one set of values to an other set of values

**INPUT (ARGUMENT, INDEPENDENT VARIABLE)**
***x***




**OUTPUT (VALUE, DEPENDENT VARIABLE)**
***f*(*x*) or *y***

**DOMAIN**

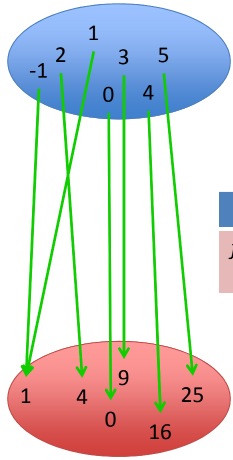



**IMAGE (RANGE)**

$x \mapsto f(x)$ or $y = f(x)$

| *x* | –1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| *f*(*x*) | 1 | 0 | 1 | 4 | 9 | 16 | 25 |

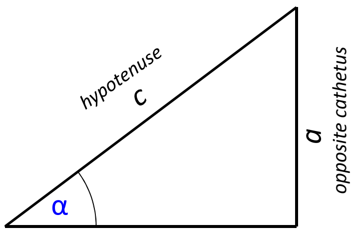
$x \mapsto f(x)$ or $y = f(x)$

*f* is the function defining the relationship between *x* and *f*(*x*)



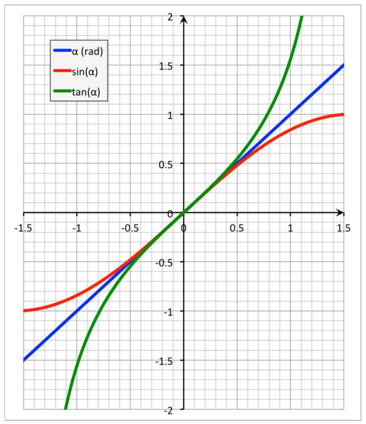
# Trigonometric Functions

**degree**: practical, traditional unit
**radian**: scientific unit, arc/radius
1 revolution = 360° = 2π rad




sine: $\sin(\alpha) = a/c$
cosine: $\cos(\alpha) = b/c$
tangent: $\tan(\alpha) = \mathrm{tg}(\alpha) = a/b$

for small angles (<10°):
$\sin(\alpha) = \alpha$ [rad] $= \tan(\alpha)$






# Linear Function

**INTEGRAL FORM**

**VARIABLES:** dependent variable — independent variable

$y = a \cdot x + b$

**PARAMETERS:** slope (gradient, increment) — y-axis intercept

**"DIFFERENTIAL" FORM**

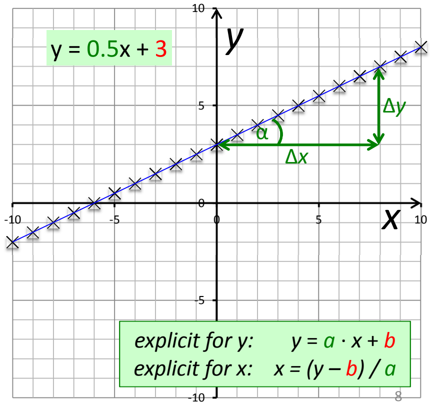
$\Delta y \propto \Delta x$

The **change** of the dependent variable is proportional to the **change** of the independent variable

if $x = 0$
then $y = b$

if $\Delta x = 1$
then $\Delta y = a$

$a = \Delta y/\Delta x = \tan\alpha$



# Linear Function: Some Examples

from the Biophysics Formula Collection

#1: The ideal gas law

(I.35)

$pV = nRT$ (if $n$ & $V$ are constant)

$p = nR/V \cdot T + 0$

$y = a \cdot x + b$

#2: Photoelectric effect

(II.37)

$E_{kin} = hf - W_{em}$

$E_{kin} = \mathbf{h} \cdot f + (-W_{em})$

$y = a \cdot x + b$

#3: Attenuation coefficient

(II.85)

$\mu = \mu_m \cdot \rho$

$\mu = \mu_m \cdot \rho + 0$

$y = a \cdot x + b$

#4: Ohm's law

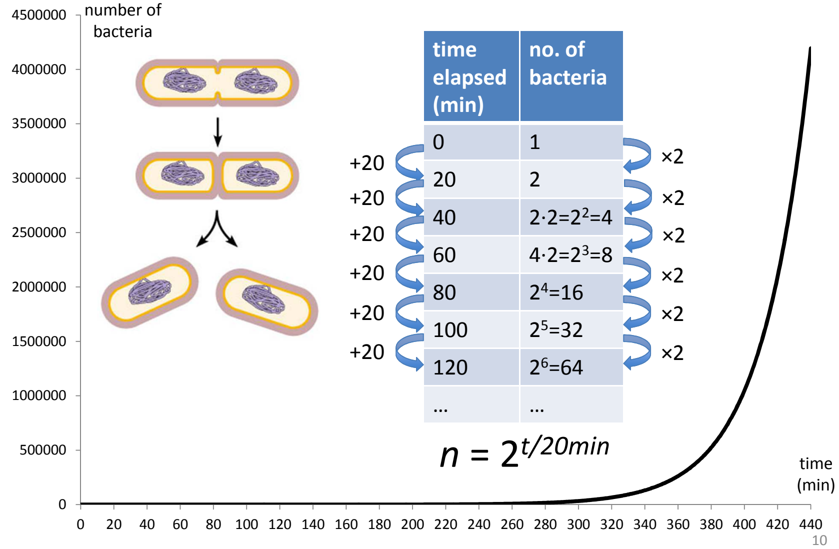
$R = U/I$

$I = 1/R \cdot U + 0$

$y = a \cdot x + b$



# Exponential Function: Example #1





| time elapsed (min) | no. of bacteria |
|---|---|
| 0 | 1 |
| 20 | 2 |
| 40 | $2 \cdot 2 = 2^2 = 4$ |
| 60 | $4 \cdot 2 = 2^3 = 8$ |
| 80 | $2^4 = 16$ |
| 100 | $2^5 = 32$ |
| 120 | $2^6 = 64$ |
| … | … |

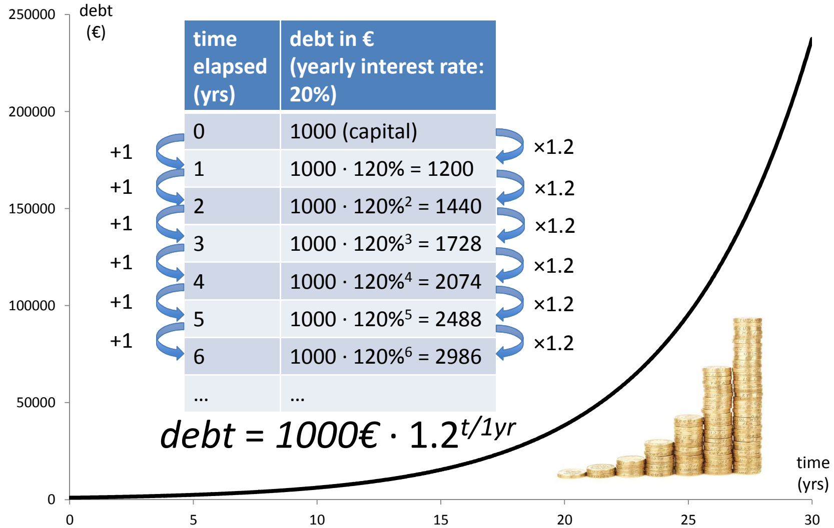
+20 +20 +20 +20 +20 +20

×2 ×2 ×2 ×2 ×2 ×2

$$n = 2^{t/20min}$$



# Exponential Function: Example #2





| time elapsed (yrs) | debt in € (yearly interest rate: 20%) |
|---|---|
| 0 | 1000 (capital) |
| 1 | $1000 \cdot 120\% = 1200$ |
| 2 | $1000 \cdot 120\%^2 = 1440$ |
| 3 | $1000 \cdot 120\%^3 = 1728$ |
| 4 | $1000 \cdot 120\%^4 = 2074$ |
| 5 | $1000 \cdot 120\%^5 = 2488$ |
| 6 | $1000 \cdot 120\%^6 = 2986$ |
| … | … |

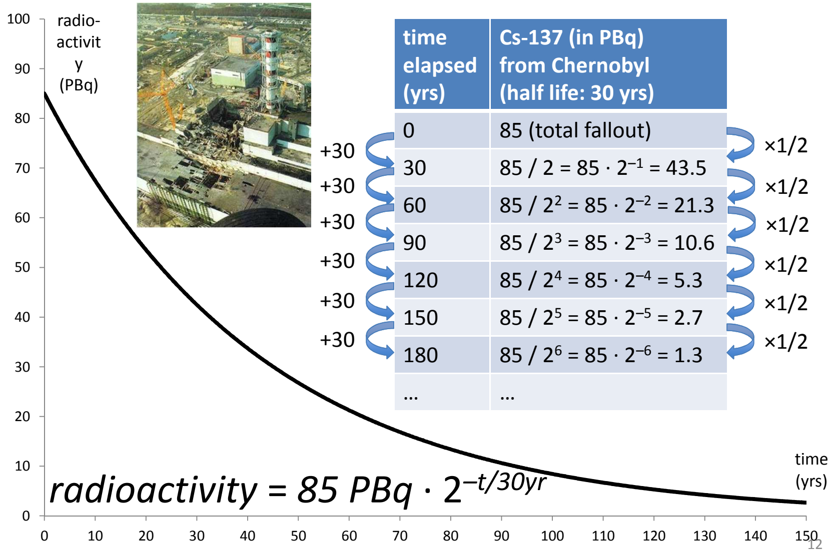
+1 +1 +1 +1 +1 +1

×1.2 ×1.2 ×1.2 ×1.2 ×1.2 ×1.2

$$debt = 1000€ \cdot 1.2^{t/1yr}$$



# Exponential Function: Example #3





| time elapsed (yrs) | Cs-137 (in PBq) from Chernobyl (half life: 30 yrs) |
|---|---|
| 0 | 85 (total fallout) |
| 30 | $85 / 2 = 85 \cdot 2^{-1} = 43.5$ |
| 60 | $85 / 2^2 = 85 \cdot 2^{-2} = 21.3$ |
| 90 | $85 / 2^3 = 85 \cdot 2^{-3} = 10.6$ |
| 120 | $85 / 2^4 = 85 \cdot 2^{-4} = 5.3$ |
| 150 | $85 / 2^5 = 85 \cdot 2^{-5} = 2.7$ |
| 180 | $85 / 2^6 = 85 \cdot 2^{-6} = 1.3$ |
| … | … |

+30 +30 +30 +30 +30 +30

×1/2 ×1/2 ×1/2 ×1/2 ×1/2 ×1/2

$$radioactivity = 85\ PBq \cdot 2^{-t/30yr}$$

# Exponential Function

INTEGRAL FORM

$$y = b \cdot a^x$$

**PRACTICAL MODIFICATIONS:**

- the base number is preferred to be $e$
- a new factor parameter $p$ (or $1/k$) is necessary in the exponent
- use a negative sign in the exponent
- $b$ is rather denoted by $y_0$

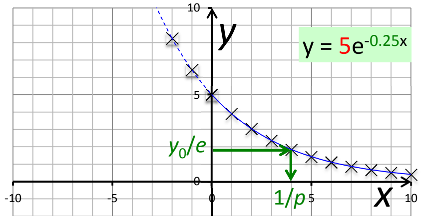
**VARIABLES:** dependent variable, independent variable

$$y = y_0 \cdot e^{-px} = y_0 \cdot e^{-x/k}$$

**PARAMETERS:** pre-exponential coefficient, exponential coefficient

if $x = 0$ then $y = y_0$

if $y = y_0/e$ then $x = 1/p = k$




*explicit for y:* $y = y_0 \cdot e^{-px}$

*explicit for x:* $x = \ln(y / y_0) / (-p)$

"DIFFERENTIAL" FORM

$$\Delta y / y \propto \Delta x$$

The **relative change** of the dependent variable is proportional to the **change** of the independent variable

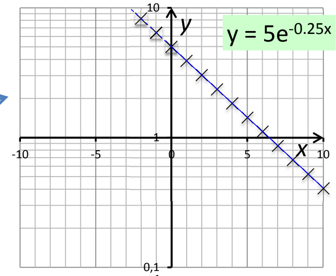


# Exponential Function: Linearization

**graphical linearization**

plot $y$ on a log scale as a function of x:

*the relationship **looks** linear but it **is** still exponential*




INTEGRAL FORM

$$y = y_0 \cdot e^{-px}$$

$$\log y = \log\left(y_0 \cdot e^{-p \cdot x}\right)$$

$$\log y = \log y_0 + \log\left(e^{-p \cdot x}\right)$$

$$\log y = \log y_0 - p \cdot x \cdot \log e$$

$$\underbrace{\log y}_{y} = \underbrace{-p \cdot \log e}_{a} \cdot x + \underbrace{\log y_0}_{b}$$

intercept = $\log y_0$

$\log 5 = 0.699$

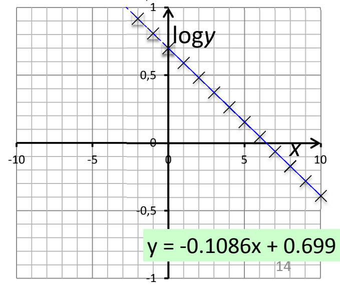
slope = $-p \cdot \log e$

$-0.25 \cdot \log e = -0.1086$

**arithmetical linearization**

plot logy as a function of $x$:

*the relationship **is** linear*






# Exponential Function: Some Examples

from the Biophysics Formula Collection

#1: Law of radiation attenuation (II.11)

$$J = J_0 \cdot e^{-\mu x}$$

$$y = y_0 \cdot e^{-px}$$

#2: Boltzmann's distribution (I.25)

$$n_i = n_0 \cdot e^{-\Delta\varepsilon/(kT)}$$

$$y = y_0 \cdot e^{-x/k}$$

#3: Decay law (II.96)

$$N = N_0 \cdot e^{-\lambda t}$$

$$y = y_0 \cdot e^{-px}$$

#4: Discharging an RC circuit (VII.2)

$$U = U_0 \cdot e^{-t/(RC)}$$

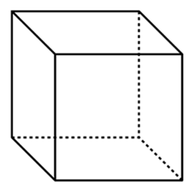
$$y = y_0 \cdot e^{-x/k}$$



# Power Function: Example

mass ∝ volume ∝ [body]length$^3$

surface area ∝ [body]length$^2$



# Power Function

**INTEGRAL FORM**

**VARIABLES:** dependent variable, independent variable

$y = b \cdot x^a$

**PARAMETERS:** pre-exponential coefficient, exponent

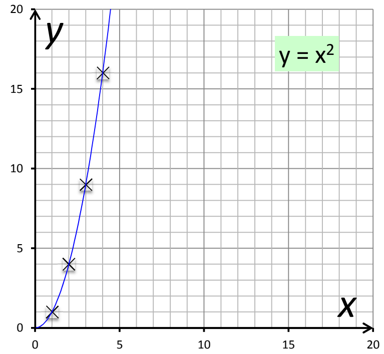
*explicit for y:* $y = b \cdot x^a$
*explicit for x:* $x = (y / b)^{1/a}$

**"DIFFERENTIAL" FORM**

$\Delta y/y \propto \Delta x/x$

The **relative change** of the dependent variable is proportional to the **relative change** of the independent variable

if $x = 1$
then $y = b$




**inverse proportionality** and **square root** functions are also power functions

$$y = \frac{b}{x} = b \cdot x^{-1}$$

$$y = \sqrt{x} = x^{\frac{1}{2}}$$



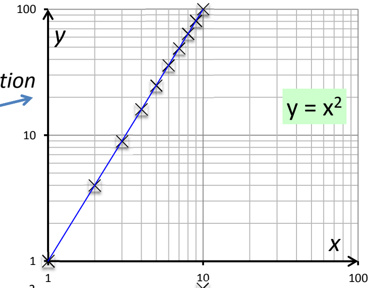
# Power Function: Linearization

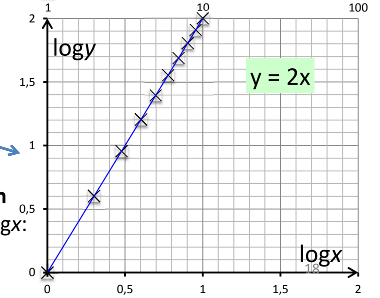
**graphical linearization**
plot both *y* and x on log scales:
*the relationship **looks** linear but it **is** still power function*

**INTEGRAL FORM**

$y = b \cdot x^a$

$$\log y = \log\left(b \cdot x^a\right)$$
$$\log y = \log b + \log\left(x^a\right)$$
$$\log y = \log b + a \cdot \log x$$
$$\underbrace{\log y}_{y} = a \cdot \underbrace{\log x}_{x} + \underbrace{\log b}_{b}$$




intercept = log*b*
log1 = 0
slope = *a*
*a* = 2

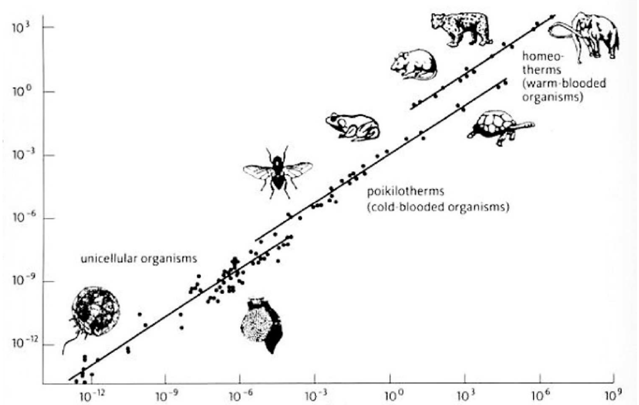
**arithmetical linearization**
plot logy as a function of log*x*:
*the relationship **is** linear*



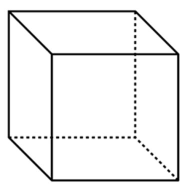
# Power Function: Example

Allometric scaling
(E.g. Kleiber's law)

hourly heat production ∝ body mass$^{3/4}$




mass ∝ volume ∝ [body]length$^3$
surface area ∝ [body]length$^2$





# Power Function: Some Examples

from the Biophysics Formula Collection

#1: The de Broglie wavelength (I.3)

$\lambda = h/p$

$\lambda = h \cdot p^{-1}$

$y = b \cdot x^a$

#2: Stefan–Boltzmann law (II.41)

$M_{black} = \sigma \cdot T^4$

$y = b \cdot x^a$

#3: Duane–Hunt law (II.80)

$$\lambda_{min} = \frac{hc}{eU_{anode}}$$

$\lambda_{min} = hc/e \cdot U^{-1}$

$y = b \cdot x^a$

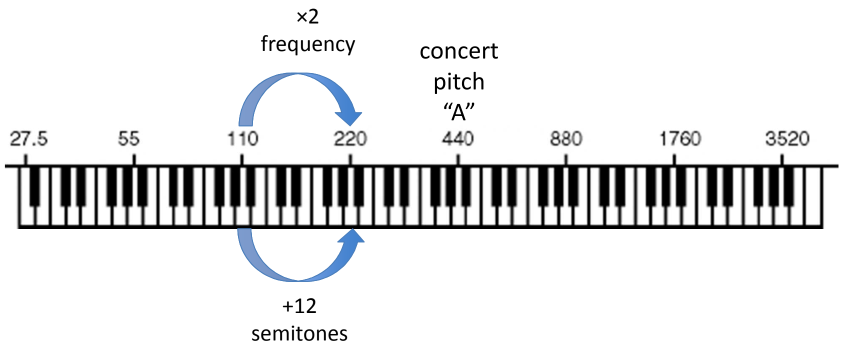
#4: Mass dependence of eigenfrequency (Resonance 6)

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$f_0 = k^{1/2}/(2\pi) \cdot m^{-1/2}$

$y = b \cdot x^a$

# Logarithmic Function: Example




# Logarithmic Function

**INTEGRAL FORM**

$$y = b \cdot \log_a(x)$$

**PRACTICAL CONSIDERATIONS:**

- base is 10 (sometimes *e* or 2)
- if the base is fixed this will modify the factor paramater according to the following identity:

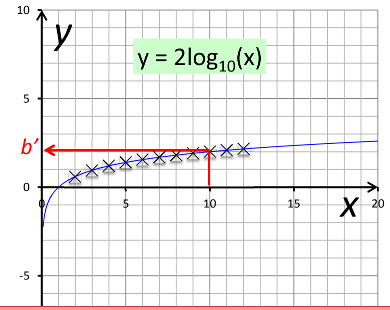
$b \cdot \log_a(x) = b/\log_{10}(a) \cdot \log_{10}(x) = b' \cdot \log_{10}(x)$

**VARIABLES:** dependent variable, independent variable

$$y = b' \cdot \log_{10}(x)$$

**PARAMETERS:** factor parameter

if $x$ = 10
then $y = b'$




**„DIFFERENTIAL” FORM**

$$\Delta y \sim \Delta x / x$$

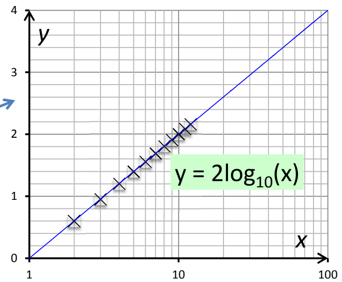
The **absolute change** of the dependent variable is proportional to the **relative change** of the independent variable

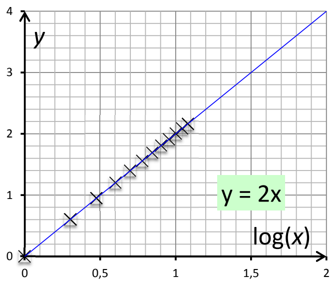
# Logarithmic Function: Linearization

**graphical linearization**
plot *y* on lin and *x* on log scales:
*the relationship **looks** linear but it **is** still a log. function*

**INTEGRAL FORM**

$$y = b' \cdot \log_{10}(x)$$




**arithmetical linearization**
plot *y* as a function of log*x*:
*the relationship **is** linear*

# Logarithmic Function: Some Examples

## from the Biophysics Formula Collection ...and elsewhere

#1: The statistical definition of entropy
(III.72)
$S = k \ln \Omega$

$$S = k \cdot \log_e(\Omega)$$
$$y = b \cdot \log_a(x)$$

#2: The decibel (dB) scale
(VII.10)
$n = 10 \log A_p$

$$n = 10 \cdot \log_{10}(A_p)$$
$$y = b \cdot \log_a(x)$$

#3: The definition of absorbance
(VI.34)
$A = \lg(J_0/J)$

$$A = 1 \cdot \log_{10}(\underbrace{J_0/J})$$
$$y = b \cdot \log_a(x)$$

#4: The pH scale

$\mathrm{pH} = -\log[\mathrm{H}^+]$

$$\mathrm{pH} = -1 \cdot \log_{10}(\underbrace{[\mathrm{H}^+]/(1\ \mathrm{M})})$$
$$y = b \cdot \log_a(x)$$

# Derivative and Integral: Example #1

Δ Δ

| x | $y = x^2$ | y' = Δy/Δx | y'' = Δ(Δy/Δx)/Δx |
|---|---|---|---|
| 0 | 0 | | |
| 1 | 1 | | |
| 2 | 4 | | |
| 3 | 9 | | |
| 4 | 16 | | |
| 5 | 25 | | |
| 6 | 36 | | |
| 7 | 49 | | |
| 8 | 64 | | |
| 9 | 81 | | |
| 10 | 100 | | |

Σ Σ



# Derivative and Integral: Example #1

Δ Δ

| x | $y = x^2$ | y' = Δy/Δx | y'' = Δ(Δy/Δx)/Δx |
|---|---|---|---|
| 0 | 0 | | |
| 1 | 1 | 1 | |
| 2 | 4 | 3 | 2 |
| 3 | 9 | 5 | 2 |
| 4 | 16 | 7 | 2 |
| 5 | 25 | 9 | 2 |
| 6 | 36 | 11 | 2 |
| 7 | 49 | 13 | 2 |
| 8 | 64 | 15 | 2 |
| 9 | 81 | 17 | 2 |
| 10 | 100 | 19 | 2 |

Σ Σ



# Derivative and Integral: Example #2

| time (s) | distance (m) | speed (m/s) | acceleration ($m/s^2$) |
|---|---|---|---|
| 0 | 0 | 0 | 10 |
| 1 | 5 | 10 | 10 |
| 2 | 20 | 20 | 10 |
| 3 | 45 | 30 | 10 |
| 4 | 80 | 40 | 10 |
| 5 | 125 | 50 | 10 |
| 6 | 180 | 60 | 10 |
| 7 | 245 | 70 | 10 |
| 8 | 320 | 80 | 10 |
| 9 | 405 | 90 | 10 |
| 10 | 500 | 100 | 10 |

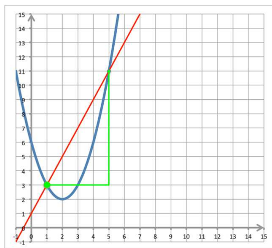


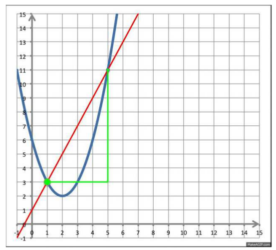
# Derivative: slope of tangent line

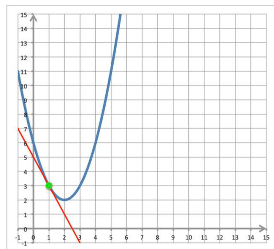
difference quotient:
Δy/Δ*x*
slope of **secant** line

$\Delta \rightarrow d$

derivative:
dy/d*x*
slope of **tangent** line

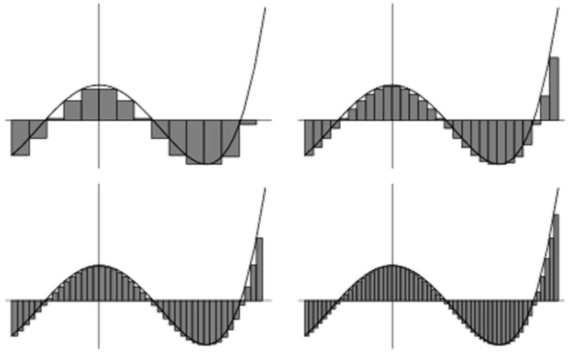






# Integral: Area Under the Curve (AUC)

$$\Sigma \longrightarrow \int$$

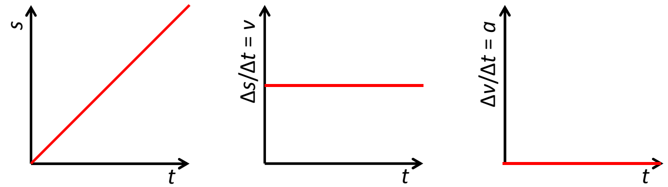




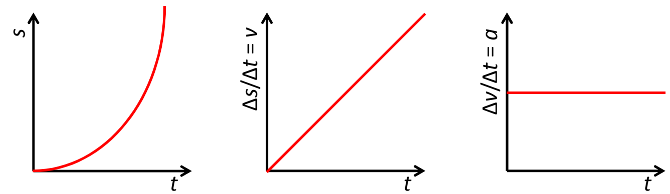
# Derivative and Integral: Application

Rectilinear Motion

uniform rectilinear motion:

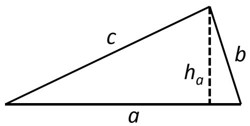



uniform rectilinear acceleration:

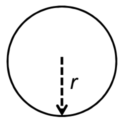




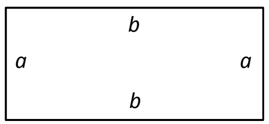

# Perimeter & Area



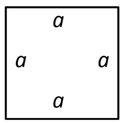

TRIANGLE
perimeter: $a+b+c$
area: $a*h_a/2$



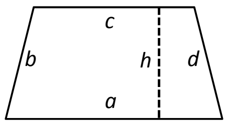

CIRCLE
perimeter: $2r\pi$
area: $r^2\pi$




RECTANGLE
perimeter: $2*(a+b)$
area: $a*b$



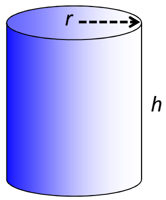

SQUARE
perimeter: $4a$
area: $a*a=a^2$



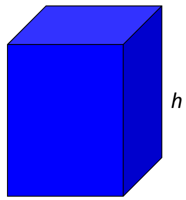

TRAPEZOID
perimeter: $a+b+c+d$
area: $(a+c)/2*h$



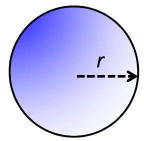
# Surface & Volume




CYLINDER (open)
surface (wall only): $2r\pi*h$
volume: $r^2\pi*h$




PRISM (open)
surface (wall only):
(perimeter of base)*$h$
volume: (area of base)*$h$




SPHERE
surface: $4r^2\pi$
volume: $4r^3\pi/3$

# Units – SI Base & Derived Units

| physical quantity | symbol | unit | symbol |
|---|---|---|---|
| length | *l, x, s, d* | meter | m |
| mass | *m* | kilogram | kg |
| time | *t* | second | s |
| temperature | *T* | kelvin | K |
| electric current | *I* | ampere | A |
| amount of substance | *n, N*, ν [nu] | mole | mol |
| luminous intensity | $I_v$ | candela | cd |

The SI base units

| physical quantity | symbol | unit | symbol | derivation |
|---|---|---|---|---|
| speed | *v, c* | – | – | $m{\cdot}s^{-1}$ |
| acceleration | *a* | – | – | $m{\cdot}s^{-2}$ |
| force | *F* | newton | N | $kg{\cdot}m{\cdot}s^{-2}$ |
| energy | *E* | joule | J | $kg{\cdot}m^2{\cdot}s^{-2}$ |
| power | *P* | watt | W | $kg{\cdot}m^2{\cdot}s^{-3}$ |
| intensity | *I* | – | – | $kg{\cdot}s^{-3}$ |
| pressure | *p* | pascal | Pa | $kg{\cdot}m^{-1}{\cdot}s^{-2}$ |

Some SI derived units



# Units – SI Prefixes

| prefix | symbol | meaning | etymology |
|---|---|---|---|
| exa | E | $\times10^{18} = \times1000^6$ | Greek 6 (ἕξ = hex) |
| peta | P | $\times10^{15} = \times1000^5$ | Greek 5 (πέντε = pente) |
| tera | T | $\times10^{12} = \times1000^4$ | Greek 4 (τέτταρες = tettares), originally: monster (τέρας = teras) |
| giga | G | $\times10^{9} = \times1000^3$ | Greek giant (γίγας = gigas) |
| mega | M | $\times10^{6} = \times1000^2$ | Greek great (μέγας = megas) |
| kilo | k | $\times10^{3} = \times1000^1$ | Greek 1000 (χίλιοι = khilioi) |
| hekto | h | $\times10^{2}$ | Greek 100 (ἑκατόν = hekaton) |
| deca | da (dk) | $\times10^{1}$ | Greek 10 (δέκα = deka) |
| | | | |
| deci | d | $\times10^{-1}$ | Latin 10 (decem) |
| centi | c | $\times10^{-2}$ | Latin 100 (centum) |
| milli | m | $\times10^{-3} = \times1000^{-1}$ | Latin 1000 (mille, *pl.* milia) |
| micro | μ | $\times10^{-6} = \times1000^{-2}$ | Greek small (μικρός = mikros) |
| nano | n | $\times10^{-9} = \times1000^{-3}$ | Greek dwarf (νᾶνος = nanos) |
| pico | p | $\times10^{-12} = \times1000^{-4}$ | Spanish small, bit (pico) |
| femto | f | $\times10^{-15} = \times1000^{-5}$ | Danish 15 (femten) |
| atto | a | $\times10^{-18} = \times1000^{-6}$ | Danish 18 (atten) |



# Units – Conversion

**from “with prefix” to “no prefix”:**
15 km = 15 · $10^3$ m
15 cg = 15 · $10^{-2}$ g

**from “no prefix” to “with prefix”:**
15 m = 15 / $10^3$ km
15 g = 15 / $10^{-2}$ cg

**from “with prefix” to “with prefix”:**
15 km = 15 · $10^3$ m = 15 · $10^3$ / $10^{-2}$ cm

**when the unit has an exponent:**
15 $km^3$ = 15 · $(10^3\ m)^3$ = 15 · $(10^3)^3$ $m^3$
15 $m^3$ = 15 / $(10^3)^3$ $km^3$

**liters to and from cubic meters:**
1 $m^3$ = 10 hL = 1000 L
1 $dm^3$ = 1 L
1 $cm^3$ = 1 mL
1 $mm^3$ = 1 μL

**time to seconds:**
2 days 3 h 12 min 30 s = ((2·24+3)·60+12)·60+30 s

**degrees, minutes of arc, seconds of arc:**
45° 40′ 30″ = $(45+40/60+30/60^2)°$

**degrees to and from radians:**
1 rad = (360/2π)°
1° = (2π/360) rad

**compound units:**
15 kg/$m^3$ = 15 · $10^3$ / (1/$(10^{-2})^3$) g/$cm^3$
45 km/h = 45 · $10^3$ / 3600 m/s

**degrees Celsius to and from kelvins:**
T = 15 °C = (15+273) K
T = 15 K = (15–273) °C
ΔT = 15 °C = 15 K
ΔT = 15 K = 15 °C