Mathematical and Physical Basis of Medical Biophysics

Lecture 1

Mathematics Necessary for Understanding Physics

Physical Quantities and Units

11th September 2017

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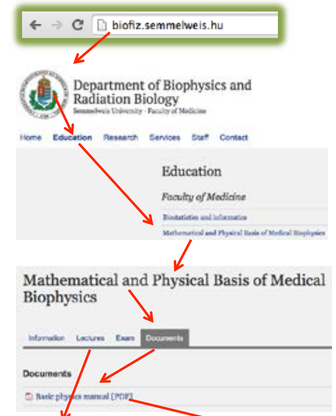
How to Get Prepared?

- university = **autonomous learning**
- sources:
  - **your** notes made in the lectures *(Monday 19^{30}–20^{50}; Friday 16^{10}–17^{30}; EOK "Szent-Györgyi Albert" lecture hall; **only in the first four weeks**)*
  - Tölgyesi: *Mathematical and Physical Basis of Medical Biophysics* (2016)
  - homepage: biofiz.semmelweis.hu
    - subject requirements
    - lecture schedule and slides
    - textbook



Lecture slides (will be uploaded one by one)

Textbook

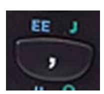
How to Use Scientific Notation?

best calculator for a medical student

natural display

still okay (but less convenient)

linear input



not allowed

programmable graphical display

Use of Symbols in Science

In science we use a large number of **Latin** and **Greek** letters (and their combinations) as symbols, so it is inevitable to learn the Greek alphabet.

However, the number of quantities and units is much greater than the number of available letters, and this can lead to confusion. Your help: CONTEXT

ambiguity! ← **BEWARE!** → **inconsistency!**

ambiguity! for example:

c and C → specific heat capacity, a general constant, Celsius, carbon, capacitance and capacitor, coulomb, centi-, concentration (many different kinds), speed of sound, speed of light

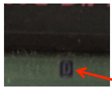
inconsistency! for example:

- density ← ***ρ*** [rho], ***d***
- speed ← ***v***, ***c***
- multiplication ← **×**, **·**, *****
- frequency ← ***f***, ***ν*** [nu]
- proportionality ← **∝**, **~**

Angles

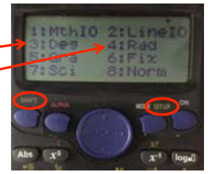
– shift
– setup
– 3 (for degrees)
– 4 (for radians)

D: degrees mode



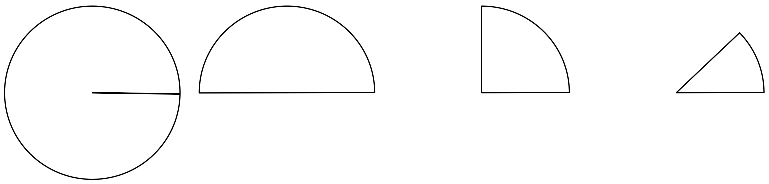
R: radians mode

revolution: one turn
degree: practical, traditional unit
radian: scientific unit, arc/radius



1 revolution = 360° = 2π rad

1° = 60′ = 3600″

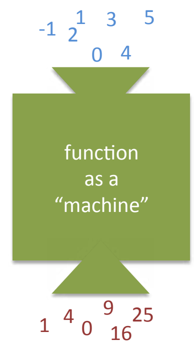


one **revolution** 360° **degrees** 2π **radians**	half **revolution** 180° **degrees** π **radian**	quarter **revolution** 90° **degrees** π/2 **radian**	1/8 **revolution** 45° **degrees** π/4 **radian**

What is a Function?

Unambiguous assignment of one set of values to an other set of values

INPUT (ARGUMENT, INDEPENDENT VARIABLE)
x



OUTPUT (VALUE, DEPENDENT VARIABLE)
f*(*x*) or *y

DOMAIN

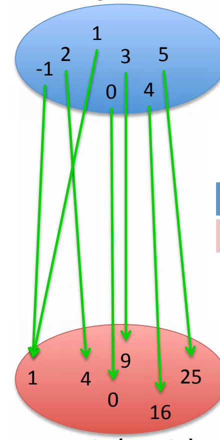


IMAGE (RANGE)

$x \mapsto f(x)$ or $y = f(x)$

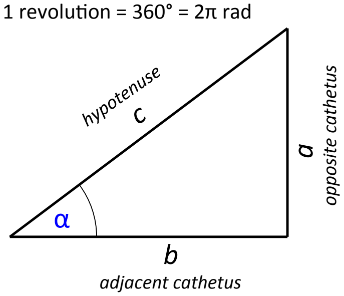
x	–1	0	1	2	3	4	5
$f(x)$	1	0	1	4	9	16	25

$x \mapsto f(x)$ or $y = f(x)$

f is the function defining the relationship between *x* and *f*(*x*)

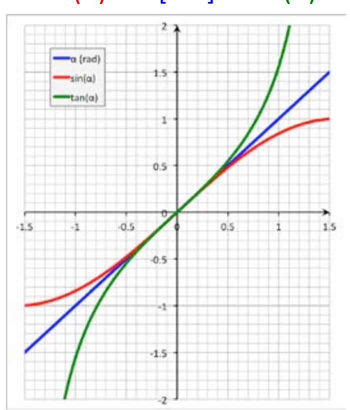
Trigonometric Functions

degree: practical, traditional unit
radian: scientific unit, arc/radius
1 revolution = 360° = 2π rad



sine: sin(α) = *a*/*c*
cosine: cos(α) = *b*/*c*
tangent: tan(α) = tg(α) = *a*/*b*

for small angles (<10° ≈ 0.2 rad):
sin(α) ≈ α [rad] ≈ tan(α)



Linear Function

INTEGRAL FORM

VARIABLES: dependent variable / independent variable

$y = a \cdot x + b$

PARAMETERS: slope (gradient, increment) / y-axis intercept

“DIFFERENTIAL” FORM

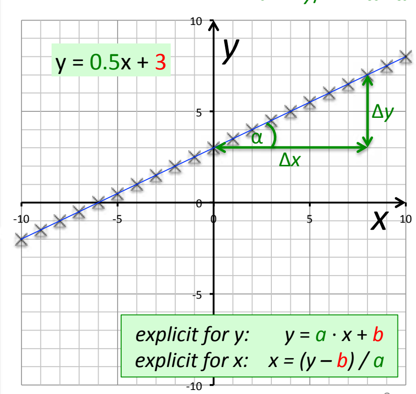
$\Delta y \propto \Delta x$

The **change** of the dependent variable is proportional to the **change** of the independent variable

if $x = 0$ then $y = b$

if $\Delta x = 1$ then $\Delta y = a$

$a = \Delta y / \Delta x = \tan\alpha$



explicit for y: $y = a \cdot x + b$
explicit for x: $x = (y - b) / a$

Linear Function: Some Examples

from the Biophysics Formula Collection

#1: The ideal gas law
(I.35)
$pV = nRT$ (if n & V are constant)

$$p = nR/V \cdot T + 0$$
$$y = a \cdot x + b$$

#2: Photoelectric effect
(II.37)
$E_{kin} = hf - W_{em}$

$$E_{kin} = \mathbf{h} \cdot f + (-W_{em})$$
$$y = a \cdot x + b$$

#3: Attenuation coefficient
(II.85)
$\mu = \mu_m \cdot \rho$

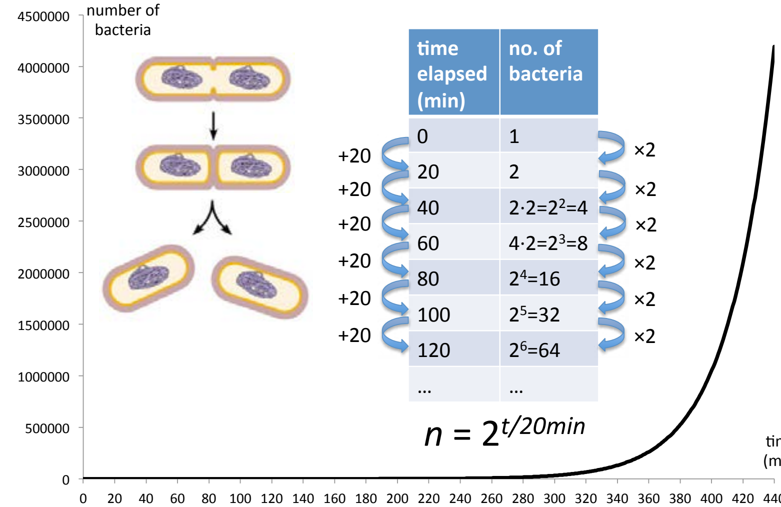
$$\mu = \mu_m \cdot \rho + 0$$
$$y = a \cdot x + b$$

#4: Ohm's law

$R = U/I$

$$I = 1/R \cdot U + 0$$
$$y = a \cdot x + b$$

Exponential Function: Example #1



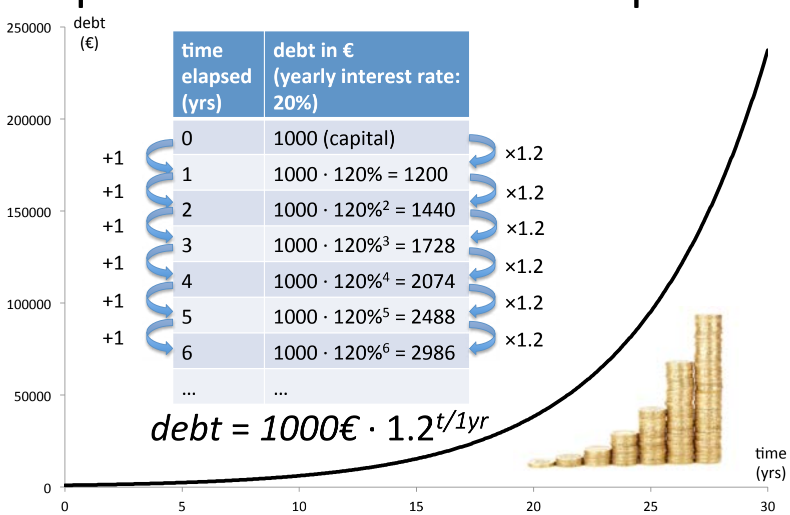
time elapsed (min)	no. of bacteria
0	1
20	2
40	$2 \cdot 2 = 2^2 = 4$
60	$4 \cdot 2 = 2^3 = 8$
80	$2^4 = 16$
100	$2^5 = 32$
120	$2^6 = 64$
...	...

+20 +20 +20 +20 +20 +20

×2 ×2 ×2 ×2 ×2 ×2

$$n = 2^{t/20min}$$

Exponential Function: Example #2



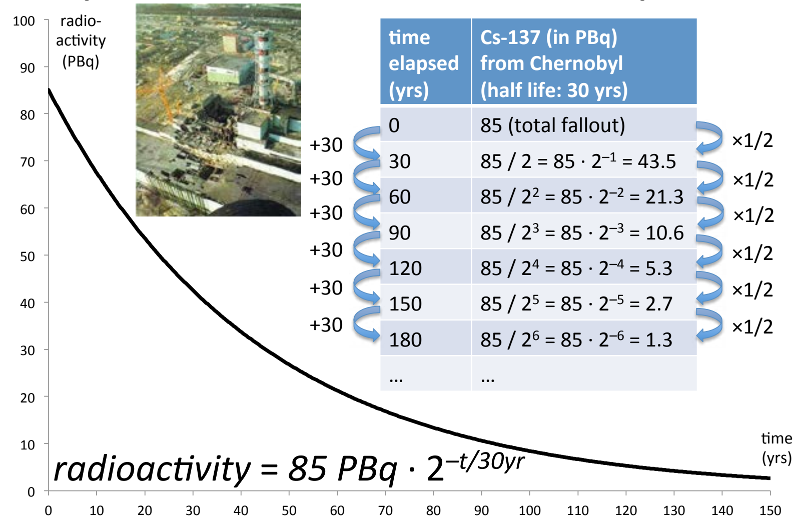
time elapsed (yrs)	debt in € (yearly interest rate: 20%)
0	1000 (capital)
1	$1000 \cdot 120\% = 1200$
2	$1000 \cdot 120\%^2 = 1440$
3	$1000 \cdot 120\%^3 = 1728$
4	$1000 \cdot 120\%^4 = 2074$
5	$1000 \cdot 120\%^5 = 2488$
6	$1000 \cdot 120\%^6 = 2986$
...	...

+1 +1 +1 +1 +1 +1

×1.2 ×1.2 ×1.2 ×1.2 ×1.2 ×1.2

$$debt = 1000€ \cdot 1.2^{t/1yr}$$

Exponential Function: Example #3



time elapsed (yrs)	Cs-137 (in PBq) from Chernobyl (half life: 30 yrs)
0	85 (total fallout)
30	$85 / 2 = 85 \cdot 2^{-1} = 43.5$
60	$85 / 2^2 = 85 \cdot 2^{-2} = 21.3$
90	$85 / 2^3 = 85 \cdot 2^{-3} = 10.6$
120	$85 / 2^4 = 85 \cdot 2^{-4} = 5.3$
150	$85 / 2^5 = 85 \cdot 2^{-5} = 2.7$
180	$85 / 2^6 = 85 \cdot 2^{-6} = 1.3$
...	...

+30 +30 +30 +30 +30 +30

×1/2 ×1/2 ×1/2 ×1/2 ×1/2 ×1/2

$$radioactivity = 85\ PBq \cdot 2^{-t/30yr}$$

Exponential Function

INTEGRAL FORM

$$y = b \cdot a^x$$

PRACTICAL MODIFICATIONS:

- the base number is preferred to be e
- a new factor parameter p (or $1/k$) is necessary in the exponent
- use a negative sign in the exponent
- b is rather denoted by y_0

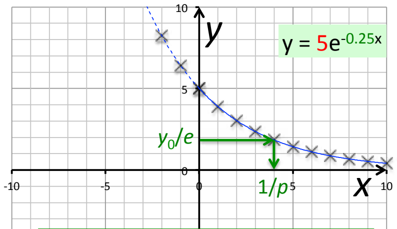
VARIABLES: dependent variable, independent variable

$$y = y_0 \cdot e^{-px} = y_0 \cdot e^{-x/k}$$

PARAMETERS: pre-exponential coefficient, exponential coefficient

if $x = 0$ then $y = y_0$

if $y = y_0/e$ then $x = 1/p = k$



explicit for y: $y = y_0 \cdot e^{-px}$

explicit for x: $x = \ln(y / y_0) / (-p)$

"DIFFERENTIAL" FORM

$$\Delta y/y \propto \Delta x$$

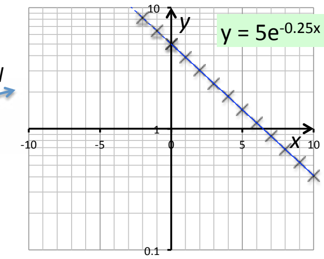
The **relative change** of the dependent variable is proportional to the **change** of the independent variable

Exponential Function: Linearization

graphical linearization

plot y on a log scale as a function of x:

*the relationship **looks** linear but it **is** still exponential*



INTEGRAL FORM

$$y = y_0 \cdot e^{-px}$$

$$\log y = \log\left(y_0 \cdot e^{-p \cdot x}\right)$$

$$\log y = \log y_0 + \log\left(e^{-p \cdot x}\right)$$

$$\log y = \log y_0 - p \cdot x \cdot \log e$$

$$\underbrace{\log y}_{y} = \underbrace{-p \cdot \log e}_{a} \cdot x + \underbrace{\log y_0}_{b}$$

intercept = $\log y_0$

$\log 5 = 0.699$

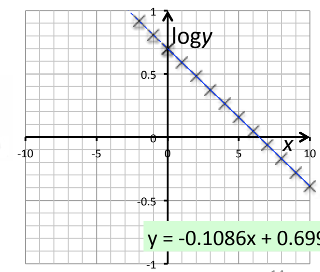
slope = $-p \cdot \log e$

$-0.25 \cdot \log e = -0.1086$

arithmetical linearization

plot $\log y$ as a function of x:

*the relationship **is** linear*



Exponential Function: Some Examples

from the Biophysics Formula Collection

#1: Law of radiation attenuation (II.11)

$$J = J_0 \cdot e^{-\mu x}$$

$$y = y_0 \cdot e^{-px}$$

#2: Boltzmann's distribution (I.25)

$$n_i = n_0 \cdot e^{-\Delta\varepsilon/(kT)}$$

$$y = y_0 \cdot e^{-x/k}$$

#3: Decay law (II.96)

$$N = N_0 \cdot e^{-\lambda t}$$

$$y = y_0 \cdot e^{-px}$$

#4: Discharging an RC circuit (VII.2)

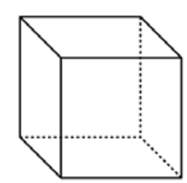
$$U = U_0 \cdot e^{-t/(RC)}$$

$$y = y_0 \cdot e^{-x/k}$$

Power Function: Example

mass $\propto$ volume $\propto$ [body]length^3

surface area $\propto$ [body]length^2



Power Function

INTEGRAL FORM

VARIABLES: dependent variable, independent variable

$y = b \cdot x^a$

PARAMETERS: pre-exponential coefficient, exponent

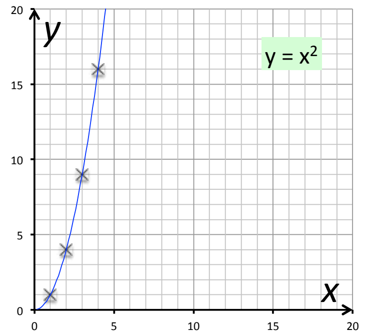
explicit for y: $y = b \cdot x^a$
explicit for x: $x = (y / b)^{1/a}$

"DIFFERENTIAL" FORM

$\Delta y/y \propto \Delta x/x$

The **relative change** of the dependent variable is proportional to the **relative change** of the independent variable

if $x = 1$
then $y = b$



inverse proportionality and **square root** functions are also power functions

$$y = \frac{b}{x} = b \cdot x^{-1}$$

$$y = \sqrt{x} = x^{\frac{1}{2}}$$

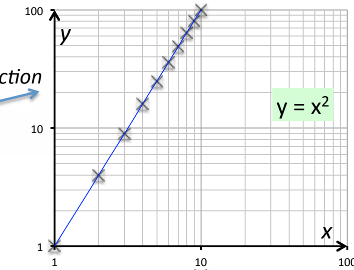
Power Function: Linearization

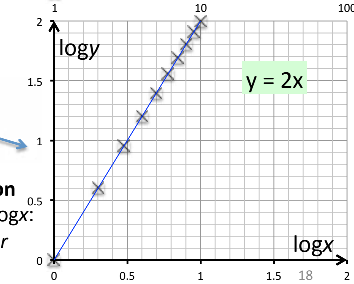
graphical linearization
plot both *y* and x on log scales:
*the relationship **looks** linear but it **is** still power function*

INTEGRAL FORM

$y = b \cdot x^a$

$$\log y = \log\left(b \cdot x^a\right)$$
$$\log y = \log b + \log\left(x^a\right)$$
$$\log y = \log b + a \cdot \log x$$
$$\underbrace{\log y}_{y} = a \cdot \underbrace{\log x}_{x} + \underbrace{\log b}_{b}$$





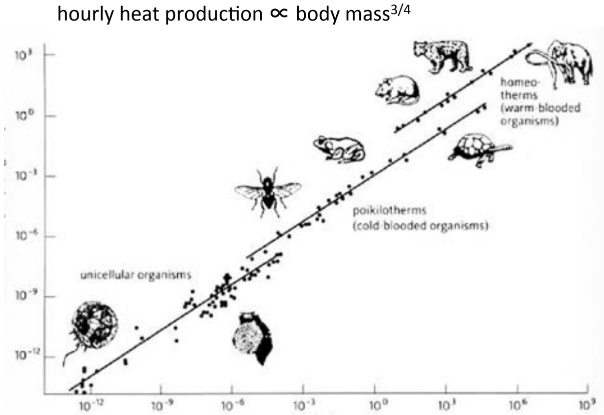
intercept = log*b*
log1 = 0
slope = *a*
a = 2

arithmetical linearization
plot logy as a function of logx:
*the relationship **is** linear*

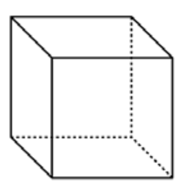
Power Function: Example

Allometric scaling
(E.g. Kleiber's law)

hourly heat production ∝ body mass$^{3/4}$



mass ∝ volume ∝ [body]length^3
surface area ∝ [body]length^2



Power Function: Some Examples
from the Biophysics Formula Collection

#1: The de Broglie wavelength
(I.3)
$\lambda = h/p$
$\lambda = h \cdot p^{-1}$
$y = b \cdot x^a$

#2: Stefan–Boltzmann law
(II.41)
$M_{black} = \sigma \cdot T^4$
$y = b \cdot x^a$

#3: Duane–Hunt law
(II.80)
$$\lambda_{min} = \frac{hc}{eU_{anode}}$$
$\lambda_{min} = hc/e \cdot U^{-1}$
$y = b \cdot x^a$

#4: Mass dependence of eigenfrequency
(Resonance 6)
$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$
$f_0 = k^{1/2}/(2\pi) \cdot m^{-1/2}$
$y = b \cdot x^a$