Mathematical and Physical Basis of Medical Biophysics

Lecture 1

Mathematics Necessary for Understanding Physics

Physical Quantities and Units

7th September 2020

Gergely AGÓCS

How to Get Prepared?

- university =
  autonomous learning
- sources:
  - **your** notes made in the lectures; ***only in the first four weeks***

G. Agócs J. Gál-Somkuti Zs. Mártonfalvi G. Schay

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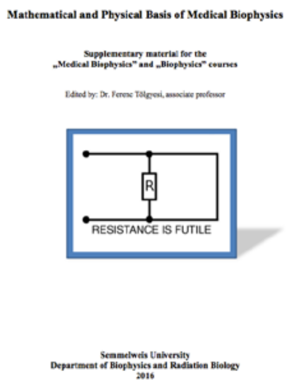
  - Tölgyesi: *Mathematical and Physical Basis of Medical Biophysics* (2016)



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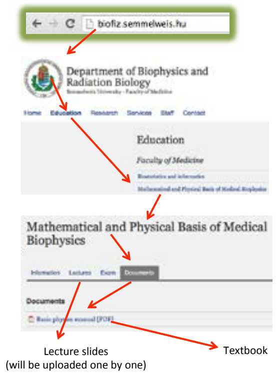
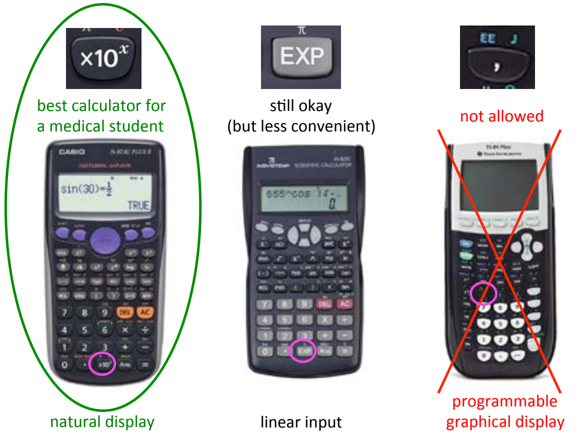
  - Tölgyesi: *Mathematical and Physical Basis of Medical Biophysics* (2016)
  - homepage: biofiz.semmelweis.hu
    - subject requirements
    - lecture schedule and slides
    - textbook



How to Use Scientific Notation?



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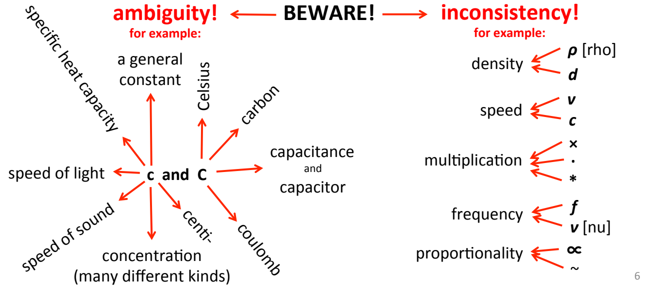
Use of Symbols in Science

In science we use a large number of **Latin** and **Greek** letters (and their combinations) as symbols, so it is inevitable to learn the Greek alphabet.

However, the number of quantities and units is much greater than the number of available letters, and this can lead to confusion. Your help: CONTEXT



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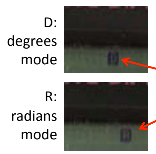
Angles



revolution: one turn

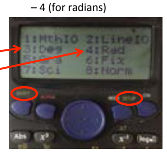
degree: practical, traditional unit

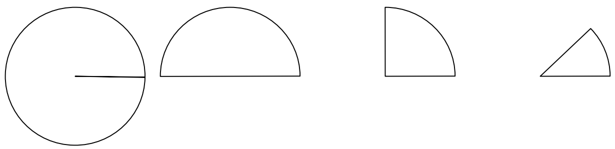
radian: scientific unit, arc/radius

1 revolution = 360° = 2π rad

1° = 60′ = 3600″

– shift
– setup
– 3 (for degrees)
– 4 (for radians)





one **revolution**	half **revolution**	quarter **revolution**	1/8 **revolution**
360° **degrees**	180° **degrees**	90° **degrees**	45° **degrees**
2π **radians**	π **radian**	π/2 **radian**	π/4 **radian**

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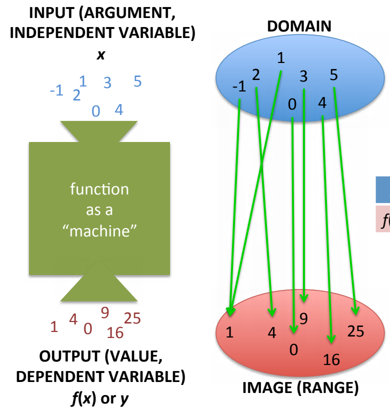
What is a Function?

Unambiguous assignment of one set of values to an other set of values



$x \mapsto f(x)$ or $y = f(x)$

x	−1	0	1	2	3	4	5
$f(x)$	1	0	1	4	9	16	25

$x \mapsto f(x)$ or $y = f(x)$

f is the function defining the relationship between *x* and *f*(*x*)

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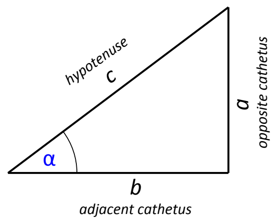
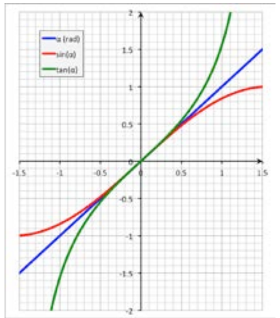
Trigonometric Functions

degree: practical, traditional unit
radian: scientific unit, arc/radius
1 revolution = 360° = 2π rad



sine: $\sin(\alpha) = a/c$
cosine: $\cos(\alpha) = b/c$
tangent: $\tan(\alpha) = \mathrm{tg}(\alpha) = a/b$

for small angles (<10° ≈ 0.2 rad):
$\sin(\alpha) \approx \alpha$ [rad] $\approx \tan(\alpha)$



Linear Function

INTEGRAL FORM

VARIABLES: dependent variable, independent variable

$y = a \cdot x + b$

PARAMETERS: slope (gradient, increment); y-axis intercept

"DIFFERENTIAL" FORM

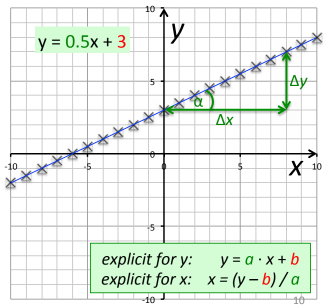
$\Delta y \propto \Delta x$

The **change** of the dependent variable is proportional to the **change** of the independent variable

if $x = 0$ then $y = b$

if $\Delta x = 1$ then $\Delta y = a$

$a = \Delta y/\Delta x = \tan\alpha$



explicit for y: $y = a \cdot x + b$
explicit for x: $x = (y - b) / a$

Linear Function: Some Examples

from the Biophysics Formula Collection

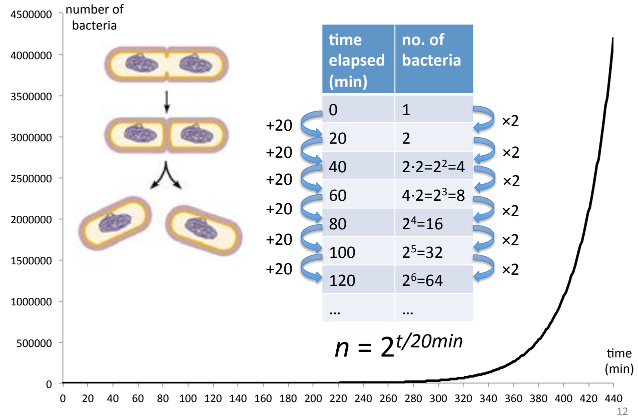
#1: The ideal gas law (I.35)
$pV = nRT$ (if n & V are constant)
$p = nR/V \cdot T + 0$
$y = a \cdot x + b$

#2: Photoelectric effect (II.37)
$E_{kin} = hf - W_{em}$
$E_{kin} = h \cdot f + (-W_{em})$
$y = a \cdot x + b$

#3: Attenuation coefficient (II.85)
$\mu = \mu_m \cdot \rho$
$\mu = \mu_m \cdot \rho + 0$
$y = a \cdot x + b$

#4: Ohm's law
$R = U/I$
$I = 1/R \cdot U + 0$
$y = a \cdot x + b$

Exponential Function: Example #1

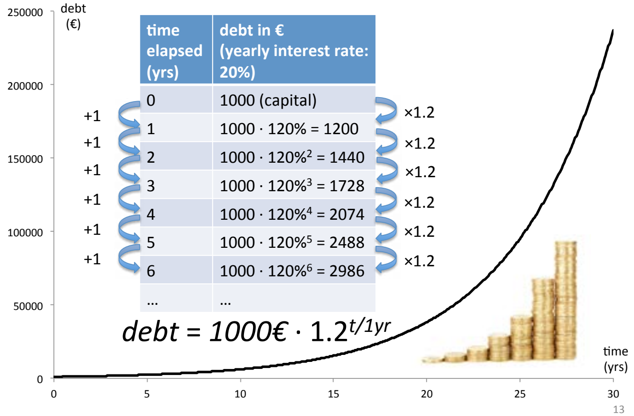


time elapsed (min)	no. of bacteria
0	1
20	2
40	$2 \cdot 2 = 2^2 = 4$
60	$4 \cdot 2 = 2^3 = 8$
80	$2^4 = 16$
100	$2^5 = 32$
120	$2^6 = 64$
…	…

+20 ×2

$n = 2^{t/20min}$

Exponential Function: Example #2



time elapsed (yrs)	debt in € (yearly interest rate: 20%)
0	1000 (capital)
1	1000 · 120% = 1200
2	$1000 \cdot 120\%^2 = 1440$
3	$1000 \cdot 120\%^3 = 1728$
4	$1000 \cdot 120\%^4 = 2074$
5	$1000 \cdot 120\%^5 = 2488$
6	$1000 \cdot 120\%^6 = 2986$
…	…

+1 (each step in time) ×1.2 (each step in debt)

$$debt = 1000€ \cdot 1.2^{t/1yr}$$

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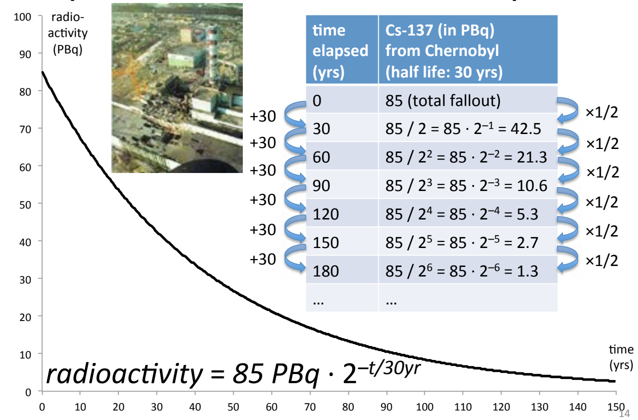
Exponential Function: Example #3



time elapsed (yrs)	Cs-137 (in PBq) from Chernobyl (half life: 30 yrs)
0	85 (total fallout)
30	$85 / 2 = 85 \cdot 2^{-1} = 42.5$
60	$85 / 2^2 = 85 \cdot 2^{-2} = 21.3$
90	$85 / 2^3 = 85 \cdot 2^{-3} = 10.6$
120	$85 / 2^4 = 85 \cdot 2^{-4} = 5.3$
150	$85 / 2^5 = 85 \cdot 2^{-5} = 2.7$
180	$85 / 2^6 = 85 \cdot 2^{-6} = 1.3$
…	…

+30 (each step in time) ×1/2 (each step in radioactivity)

$$radioactivity = 85\ PBq \cdot 2^{-t/30yr}$$

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Exponential Function

INTEGRAL FORM

$$y = b \cdot a^x$$

PRACTICAL MODIFICATIONS:

- the base number is preferred to be *e*
- a new factor parameter *p* (or 1/*k*) is necessary in the exponent
- use a negative sign in the exponent
- *b* is rather denoted by y_0

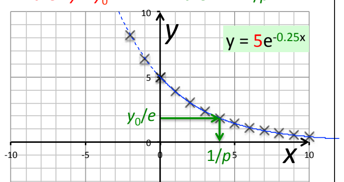
VARIABLES: dependent variable, independent variable

$$y = y_0 \cdot e^{-px} = y_0 \cdot e^{-x/k}$$

PARAMETERS: pre-exponential coefficient, exponential coefficient

if $x = 0$ then $y = y_0$

if $y = y_0/e$ then $x = 1/p = k$



explicit for y: $y = y_0 \cdot e^{-px}$

explicit for x: $x = \ln(y / y_0) / (-p)$

"DIFFERENTIAL" FORM

$$\Delta y/y \propto \Delta x$$

The **relative change** of the dependent variable is proportional to the **change** of the independent variable

Exponential Function: Linearization

graphical linearization

plot *y* on a log scale as a function of x:

the relationship ***looks*** *linear but it* ***is*** *still exponential*

INTEGRAL FORM

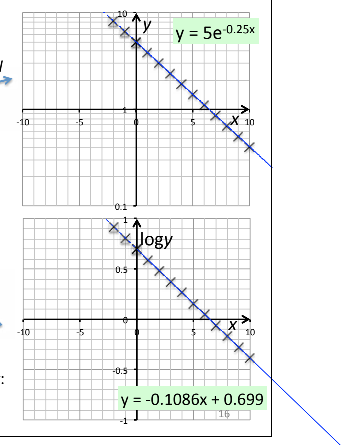
$$y = y_0 \cdot e^{-px}$$

$$\log y = \log\left(y_0 \cdot e^{-p \cdot x}\right)$$

$$\log y = \log y_0 + \log\left(e^{-p \cdot x}\right)$$

$$\log y = \log y_0 - p \cdot x \cdot \log e$$

$$\underbrace{\log y}_{y} = \underbrace{-p \cdot \log e}_{a} \cdot x + \underbrace{\log y_0}_{b}$$



intercept = $\log(y_0)$

$\log(5) = 0.699$

slope = $-p \cdot \log(e)$

$-0.25 \cdot \log(e) = -0.1086$

arithmetical linearization

plot log(*y*) as a function of *x*:

the relationship ***is*** *linear*

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Exponential Function: Some Examples

from the Biophysics Formula Collection

#1: Law of radiation attenuation (II.11)

$$J = J_0 \cdot e^{-\mu x}$$
$$y = y_0 \cdot e^{-px}$$

#2: Boltzmann's distribution (I.25)

$$n_i = n_0 \cdot e^{-\Delta\varepsilon/(kT)}$$
$$y = y_0 \cdot e^{-x/k}$$

#3: Decay law (II.96)

$$N = N_0 \cdot e^{-\lambda t}$$
$$y = y_0 \cdot e^{-px}$$

#4: Discharging an RC circuit (VII.2)

$$U = U_0 \cdot e^{-t/(RC)}$$
$$y = y_0 \cdot e^{-x/k}$$

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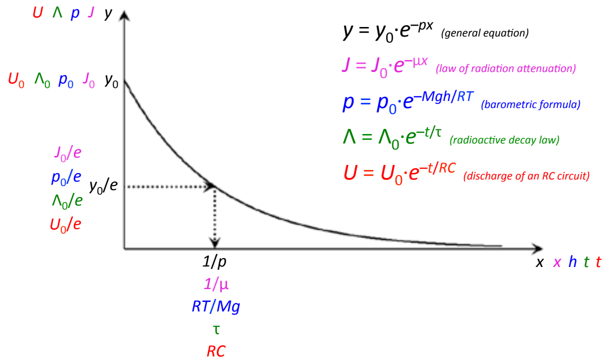
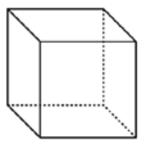
Graph of Exponential Functions

from the Biophysics Formula Collection



Power Function: Example

mass ∝ volume ∝ [body]length^3
surface area ∝ [body]length^2



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Power Function

INTEGRAL FORM

VARIABLES: dependent variable, independent variable

$$y = b \cdot x^{a}$$

PARAMETERS: pre-exponential coefficient; exponent

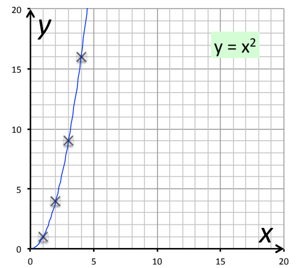
explicit for y: $y = b \cdot x^{a}$
explicit for x: $x = (y / b)^{1/a}$

"DIFFERENTIAL" FORM

$$\Delta y / y \propto \Delta x / x$$

The **relative change** of the dependent variable is proportional to the **relative change** of the independent variable

if $x = 1$
then $y = b$



inverse proportionality and **square root** functions are also power functions

$$y = \frac{b}{x} = b \cdot x^{-1}$$
$$y = \sqrt{x} = x^{\frac{1}{2}}$$

Power Function: Linearization

graphical linearization
plot both *y* and x on log scales:
*the relationship **looks** linear but it **is** still power function*

INTEGRAL FORM

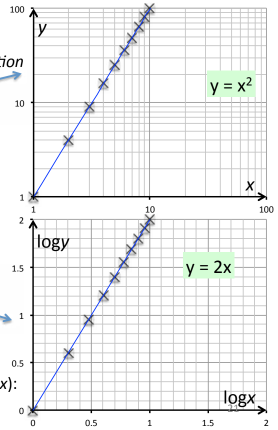
$y = b \cdot x^a$

$\log y = \log(b \cdot x^a)$

$\log y = \log b + \log(x^a)$

$\log y = \log b + a \cdot \log x$

$\underbrace{\log y}_{y} = a \cdot \underbrace{\log x}_{x} + \underbrace{\log b}_{b}$



intercept = log*b*
log1 = 0
slope = *a*
a = 2

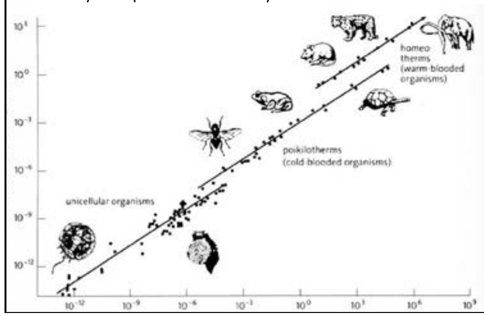
arithmetical linearization
plot log(*y*) as a function of log(*x*):
*the relationship **is** linear*

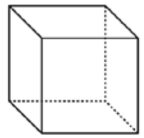
Power Function: Example

Allometric scaling
(E.g. Kleiber's law)

mass ∝ volume ∝ [body]length^3
surface area ∝ [body]length^2

hourly heat production ∝ body mass$^{3/4}$





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Power Function: Some Examples
from the Biophysics Formula Collection

#1: The de Broglie wavelength
(I.3)
$\lambda = h/p$

$\lambda = h \cdot p^{-1}$

$y = b \cdot x^a$

#2: Stefan–Boltzmann law
(II.41)

$M_{black} = \sigma \cdot T^4$

$y = b \cdot x^a$

#3: Duane–Hunt law
(II.80)

$\lambda_{min} = \frac{hc}{eU_{anode}}$

$\lambda_{min} = hc/e \cdot U^{-1}$

$y = b \cdot x^a$

#4: Mass dependence of eigenfrequency
(Resonance 6)

$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

$f_0 = k^{1/2}/(2\pi) \cdot m^{-1/2}$

$y = b \cdot x^a$

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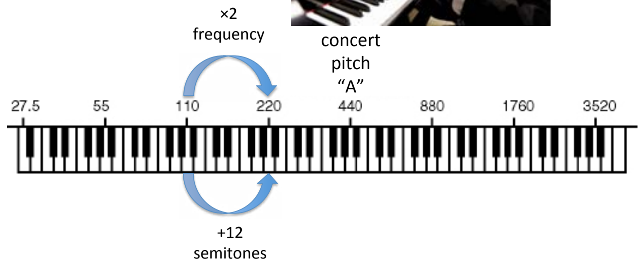
Logarithmic Function: Example



Logarithmic Function

INTEGRAL FORM

$$y = b \cdot \log_a(x)$$

PRACTICAL CONSIDERATIONS:

- base is 10 (sometimes *e* or 2)
- if the base is fixed this will modify the factor paramater according to the following identity:

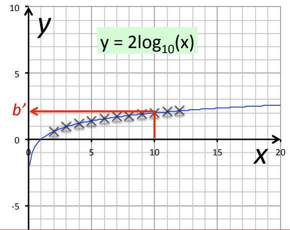
$b \cdot \log_a(x) = b/\log_{10}(a) \cdot \log_{10}(x) = b' \cdot \log_{10}(x)$

VARIABLES: dependent variable, independent variable

$$y = b' \cdot \log_{10}(x)$$

PARAMETERS: factor parameter

if $x = 10$
then $y = b'$



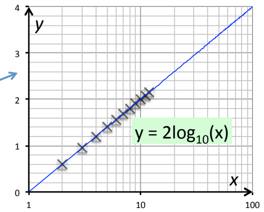
„DIFFERENTIAL" FORM

$$\Delta y \sim \Delta x/x$$

The **absolute change** of the dependent variable is proportional to the **relative change** of the independent variable

Logarithmic Function: Linearization

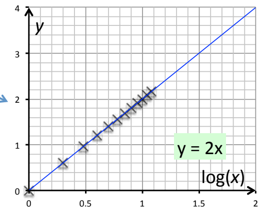
graphical linearization
plot *y* on lin and x on log scales:
*the relationship **looks** linear but it **is** still a log function*



INTEGRAL FORM

$$y = b' \cdot \log_{10}(x)$$

arithmetical linearization
plot *y* as a function of log(*x*):
*the relationship **is** linear*



Logarithmic Function: Some Examples

from the Biophysics Formula Collection ...and elsewhere

#1: The statistical definition of entropy (III.72)

$S = k \ln \Omega$

$$S = k \cdot \log_e(\Omega)$$
$$y = b \cdot \log_a(x)$$

#2: The decibel (dB) scale (VII.10)

$n = 10 \log A_p$

$$n = 10 \cdot \log_{10}(A_p)$$
$$y = b \cdot \log_a(x)$$

#3: The definition of absorbance (VI.34)

$A = \log(J_0/J)$

$$A = 1 \cdot \log_{10}(\underbrace{J_0/J})$$
$$y = b \cdot \log_a(x)$$

#4: The pH scale

$pH = -\log[H^+]$

$$pH = -1 \cdot \log_{10}(\underbrace{[H^+]/(1\ M)})$$
$$y = b \cdot \log_a(x)$$

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Functions

Summary

LINEAR FUNCTION	**EXPONENTIAL FUNCTION**
$\Delta y \sim \Delta x$	$\Delta y/y \sim \Delta x$
The **absolute change** of the dependent variable is proportional to the **absolute change** of the independent variable	The **relative change** of the dependent variable is proportional to the **absolute change** of the independent variable
y* vs. *x	**log*y* vs. *x***

Linearization

y* vs. log*x	**log*y* vs. log*x***
LOGARITHMIC FUNCTION	**POWER FUNCTION**
$\Delta y \sim \Delta x/x$	$\Delta y/y \sim \Delta x/x$
The **absolute change** of the dependent variable is proportional to the **relative change** of the independent variable	The **relative change** of the dependent variable is proportional to the **relative change** of the independent variable

Derivative and Integral: Example #1

Δ Δ

x	y = x^2	y' = Δy/Δx	y'' = Δ(Δy/Δx)/Δx
0	0		
1	1		
2	4		
3	9		
4	16		
5	25		
6	36		
7	49		
8	64		
9	81		
10	100		

Σ Σ

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Derivative and Integral: Example #1

Δ Δ

x	y = x^2	y' = Δy/Δx	y'' = Δ(Δy/Δx)/Δx
0	0		
1	1	1	
2	4	3	2
3	9	5	2
4	16	7	2
5	25	9	2
6	36	11	2
7	49	13	2
8	64	15	2
9	81	17	2
10	100	19	2

Σ Σ

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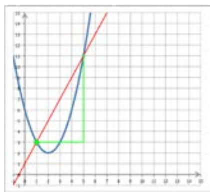
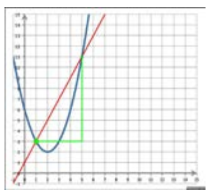
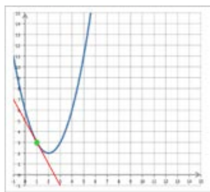
Derivative: slope of tangent line

difference quotient:
Δy/Δx
slope of **secant** line

$$\Delta \longrightarrow \mathrm{d}$$

derivative:
dy/dx
slope of **tangent** line

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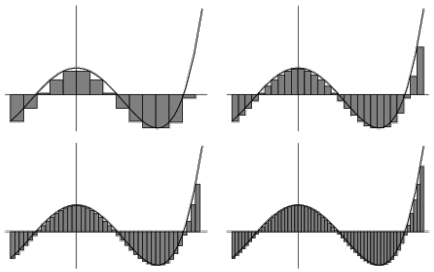
Integral: Area Under the Curve (AUC)

$$\Sigma \longrightarrow \int$$



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