Mathematical and Physical Basis of Medical Biophysics

Lecture 1

Mathematics Necessary for Understanding Physics

Physical Quantities and Units

11th September 2020

Gergely AGÓCS

How to Get Prepared?

- university = **autonomous learning**
- sources:
  - **your** notes made in the lectures; ***only in the first four weeks***)

G. Agócs J. Gál-Somkuti Zs. Mártonfalvi G. Schay

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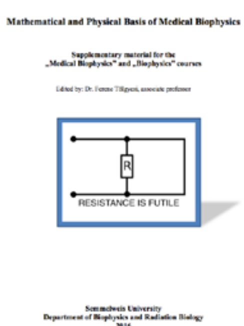
  - Tölgyesi: *Mathematical and Physical Basis of Medical Biophysics* (2016)



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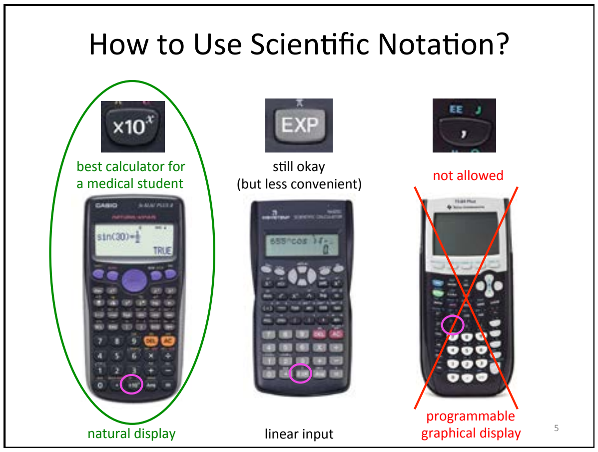
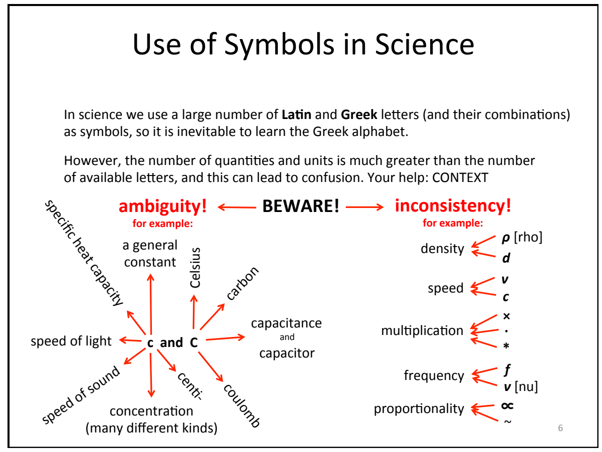
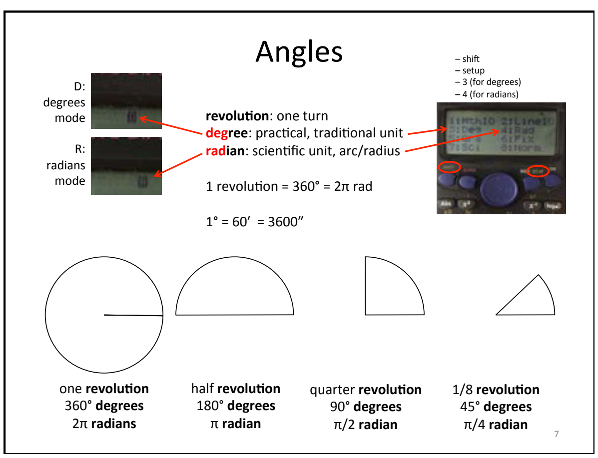
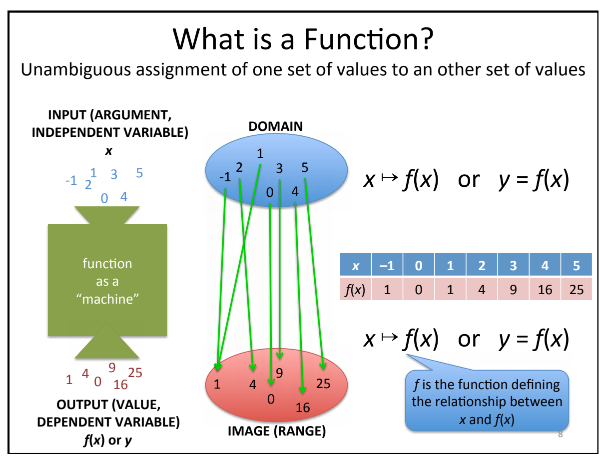
  - Tölgyesi: *Mathematical and Physical Basis of Medical Biophysics* (2016)
  - homepage: biofiz.semmelweis.hu
    - subject requirements
    - lecture schedule and slides
    - textbook


How to Use Scientific Notation?
x10^x
EXP
best calculator for
a medical student
still okay
(but less convenient)
not allowed
natural display
linear input
programmable
graphical display
5


Use of Symbols in Science
In science we use a large number of Latin and Greek letters (and their combinations) as symbols, so it is inevitable to learn the Greek alphabet.
However, the number of quantities and units is much greater than the number of available letters, and this can lead to confusion. Your help: CONTEXT
ambiguity!
BEWARE!
inconsistency!
for example:
for example:
specific heat capacity
a general constant
Celsius
carbon
speed of light
c and C
capacitance and capacitor
speed of sound
centi-
coulomb
concentration (many different kinds)
density ρ [rho] d
speed v c
multiplication × · *
frequency f ν [nu]
proportionality ∝ ~
6


Angles
D: degrees mode
R: radians mode
– shift
– setup
– 3 (for degrees)
– 4 (for radians)
revolution: one turn
degree: practical, traditional unit
radian: scientific unit, arc/radius
1 revolution = 360° = 2π rad
1° = 60′ = 3600″
one revolution
360° degrees
2π radians
half revolution
180° degrees
π radian
quarter revolution
90° degrees
π/2 radian
1/8 revolution
45° degrees
π/4 radian
7


What is a Function?
Unambiguous assignment of one set of values to an other set of values
INPUT (ARGUMENT, INDEPENDENT VARIABLE)
x
DOMAIN
function as a "machine"
x ↦ f(x) or y = f(x)
x | −1 | 0 | 1 | 2 | 3 | 4 | 5
f(x) | 1 | 0 | 1 | 4 | 9 | 16 | 25
x ↦ f(x) or y = f(x)
f is the function defining the relationship between x and f(x)
OUTPUT (VALUE, DEPENDENT VARIABLE)
f(x) or y
IMAGE (RANGE)
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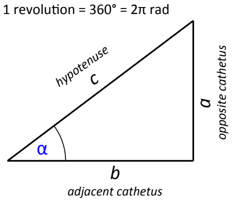
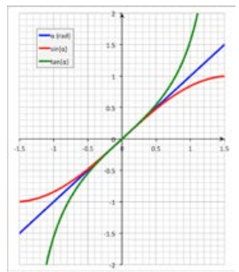
Trigonometric Functions

degree: practical, traditional unit
radian: scientific unit, arc/radius
1 revolution = 360° = 2π rad



sine: $\sin(\alpha) = a/c$
cosine: $\cos(\alpha) = b/c$
tangent: $\tan(\alpha) = \mathrm{tg}(\alpha) = a/b$

for small angles (<10° ≈ 0.2 rad):
$\sin(\alpha) \approx \alpha\ [\mathrm{rad}] \approx \tan(\alpha)$



Linear Function

INTEGRAL FORM

VARIABLES: dependent variable, independent variable

$y = a \cdot x + b$

PARAMETERS: slope (gradient, increment), y-axis intercept

"DIFFERENTIAL" FORM

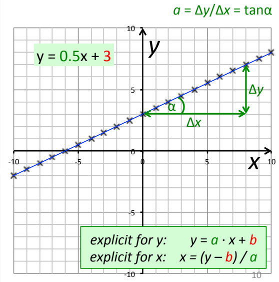
$\Delta y \propto \Delta x$

The **change** of the dependent variable is proportional to the **change** of the independent variable

if $x = 0$ then $y = b$

if $\Delta x = 1$ then $\Delta y = a$

$a = \Delta y/\Delta x = \tan\alpha$



explicit for y: $y = a \cdot x + b$
explicit for x: $x = (y - b) / a$

Linear Function: Some Examples
from the Biophysics Formula Collection

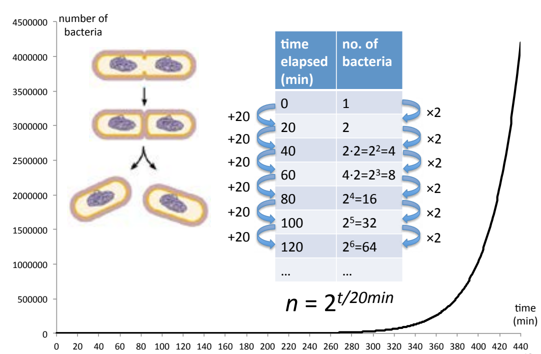
#1: The ideal gas law
(I.35)
$pV = nRT$ (if n & V are constant)
$p = nR/V \cdot T + 0$
$y = a \cdot x + b$

#2: Photoelectric effect
(II.37)
$E_{kin} = hf - W_{em}$
$E_{kin} = \boldsymbol{h} \cdot f + (-W_{em})$
$y = a \cdot x + b$

#3: Attenuation coefficient
(II.85)
$\mu = \mu_m \cdot \rho$
$\mu = \mu_m \cdot \rho + 0$
$y = a \cdot x + b$

#4: Ohm's law
$R = U/I$
$I = 1/R \cdot U + 0$
$y = a \cdot x + b$

Exponential Function: Example #1

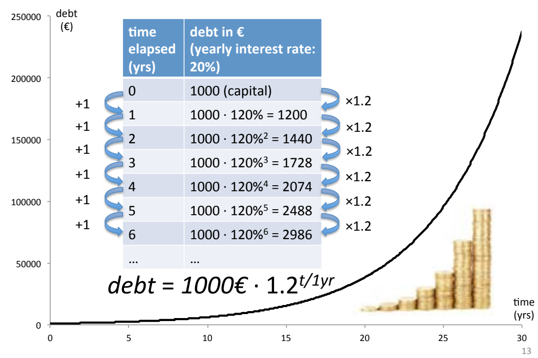


time elapsed (min)	no. of bacteria
0	1
20	2
40	$2 \cdot 2 = 2^2 = 4$
60	$4 \cdot 2 = 2^3 = 8$
80	$2^4 = 16$
100	$2^5 = 32$
120	$2^6 = 64$
...	...

+20 (between successive times); ×2 (between successive counts)

$n = 2^{t/20\mathrm{min}}$

Exponential Function: Example #2



time elapsed (yrs)	debt in € (yearly interest rate: 20%)
0	1000 (capital)
1	1000 · 120% = 1200
2	1000 · 120%² = 1440
3	1000 · 120%³ = 1728
4	1000 · 120%⁴ = 2074
5	1000 · 120%⁵ = 2488
6	1000 · 120%⁶ = 2986
…	…

$debt = 1000€ \cdot 1.2^{t/1yr}$

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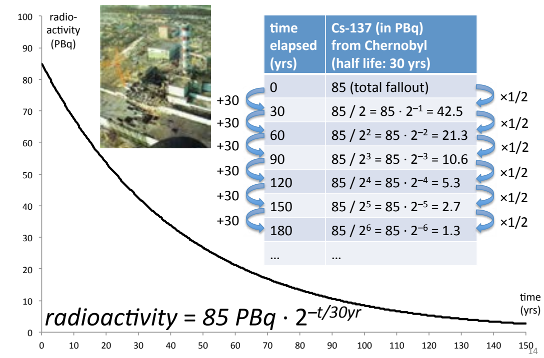
Exponential Function: Example #3



time elapsed (yrs)	Cs-137 (in PBq) from Chernobyl (half life: 30 yrs)
0	85 (total fallout)
30	$85 / 2 = 85 \cdot 2^{-1} = 42.5$
60	$85 / 2^2 = 85 \cdot 2^{-2} = 21.3$
90	$85 / 2^3 = 85 \cdot 2^{-3} = 10.6$
120	$85 / 2^4 = 85 \cdot 2^{-4} = 5.3$
150	$85 / 2^5 = 85 \cdot 2^{-5} = 2.7$
180	$85 / 2^6 = 85 \cdot 2^{-6} = 1.3$
…	…

$radioactivity = 85\ PBq \cdot 2^{-t/30yr}$

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Exponential Function

INTEGRAL FORM

$$y = b \cdot a^x$$

PRACTICAL MODIFICATIONS:

- the base number is preferred to be e
- a new factor parameter p (or $1/k$) is necessary in the exponent
- use a negative sign in the exponent
- b is rather denoted by y_0

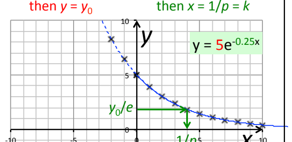
VARIABLES: dependent variable, independent variable

$$y = y_0 \cdot e^{-px} = y_0 \cdot e^{-x/k}$$

PARAMETERS: pre-exponential coefficient, exponential coefficient

if $x = 0$ then $y = y_0$

if $y = y_0/e$ then $x = 1/p = k$



explicit for y: $y = y_0 \cdot e^{-px}$

explicit for x: $x = \ln(y / y_0) / (-p)$

"DIFFERENTIAL" FORM

$$\Delta y/y \propto \Delta x$$

The **relative change** of the dependent variable is proportional to the **change** of the independent variable

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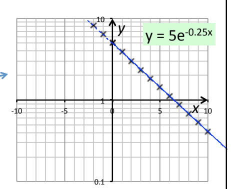
Exponential Function: Linearization

graphical linearization

plot y on a log scale as a function of x:

*the relationship **looks** linear but it **is** still exponential*



INTEGRAL FORM

$$y = y_0 \cdot e^{-px}$$

$$\log y = \log\left(y_0 \cdot e^{-p \cdot x}\right)$$

$$\log y = \log y_0 + \log\left(e^{-p \cdot x}\right)$$

$$\log y = \log y_0 - p \cdot x \cdot \log e$$

$$\underbrace{\log y}_{y} = \underbrace{-p \cdot \log e}_{a} \cdot x + \underbrace{\log y_0}_{b}$$

intercept = $\log(y_0)$

$\log(5) = 0.699$

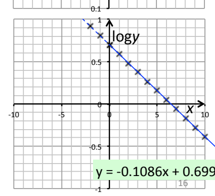
slope = $-p \cdot \log(e)$

$-0.25 \cdot \log(e) = -0.1086$

arithmetical linearization

plot $\log(y)$ as a function of x:

*the relationship **is** linear*



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Exponential Function: Some Examples

from the Biophysics Formula Collection

#1: Law of radiation attenuation (II.11)

$J = J_0 \cdot e^{-\mu x}$

$y = y_0 \cdot e^{-px}$

#2: Boltzmann's distribution (I.25)

$n_i = n_0 \cdot e^{-\Delta\varepsilon/(kT)}$

$y = y_0 \cdot e^{-x/k}$

#3: Decay law (II.96)

$N = N_0 \cdot e^{-\lambda t}$

$y = y_0 \cdot e^{-px}$

#4: Discharging an RC circuit (VII.2)

$U = U_0 \cdot e^{-t/(RC)}$

$y = y_0 \cdot e^{-x/k}$

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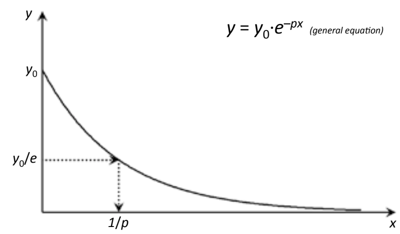
Graph of Exponential Functions

from the Biophysics Formula Collection



Graph of Exponential Functions

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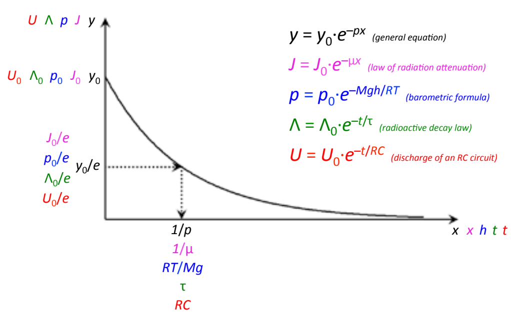


$y = y_0 \cdot e^{-px}$ (general equation)

$J = J_0 \cdot e^{-\mu x}$ (law of radiation attenuation)

$p = p_0 \cdot e^{-Mgh/RT}$ (barometric formula)

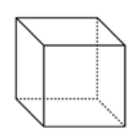
$\Lambda = \Lambda_0 \cdot e^{-t/\tau}$ (radioactive decay law)

$U = U_0 \cdot e^{-t/RC}$ (discharge of an RC circuit)

Power Function: Example

mass ∝ volume ∝ $[\text{body}]\text{length}^3$

surface area ∝ $[\text{body}]\text{length}^2$



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Power Function

INTEGRAL FORM

VARIABLES: dependent variable — independent variable

$$y = b \cdot x^{a}$$

PARAMETERS: pre-exponential coefficient — exponent

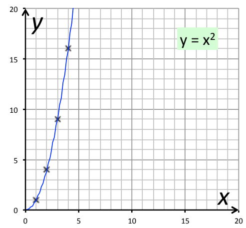
explicit for y: $y = b \cdot x^{a}$
explicit for x: $x = (y / b)^{1/a}$

"DIFFERENTIAL" FORM

$$\Delta y/y \propto \Delta x/x$$

The **relative change** of the dependent variable is proportional to the **relative change** of the independent variable

if $x = 1$
then $y = b$



inverse proportionality and **square root** functions are also power functions

$$y = \frac{b}{x} = b \cdot x^{-1}$$

$$y = \sqrt{x} = x^{\frac{1}{2}}$$

Power Function: Linearization

graphical linearization
plot both y and x on log scales:
*the relationship **looks** linear but it **is** still power function*

INTEGRAL FORM

$$y = b \cdot x^{a}$$

$$\log y = \log\left(b \cdot x^{a}\right)$$

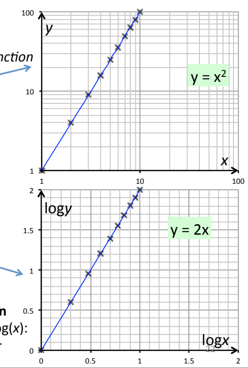
$$\log y = \log b + \log\left(x^{a}\right)$$

$$\log y = \log b + a \cdot \log x$$

$$\underbrace{\log y}_{y} = a \cdot \underbrace{\log x}_{x} + \underbrace{\log b}_{b}$$

intercept = log*b*
log1 = 0
slope = *a*
a = 2

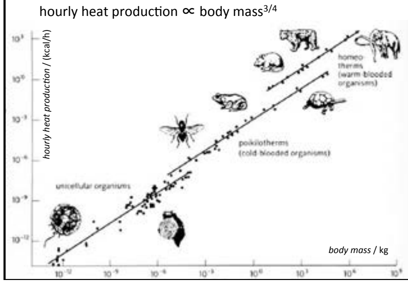
arithmetical linearization
plot log(y) as a function of log(x):
*the relationship **is** linear*



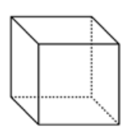
Power Function: an Example

Allometric scaling
(E.g. Kleiber's law)

hourly heat production ∝ body mass$^{3/4}$



mass ∝ volume ∝ [body]length^{3}
surface area ∝ [body]length^{2}



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Power Function: Some Examples

from the Biophysics Formula Collection

#1: The de Broglie wavelength
(I.3)
$\lambda = h/p$

$$\lambda = h \cdot p^{-1}$$
$$y = b \cdot x^{a}$$

#2: Stefan–Boltzmann law
(II.41)

$$M_{\text{black}} = \sigma \cdot T^{4}$$
$$y = b \cdot x^{a}$$

#3: Duane–Hunt law
(II.80)

$$\lambda_{\min} = \frac{hc}{eU_{\text{anode}}}$$

$$\lambda_{\min} = hc/e \cdot U^{-1}$$
$$y = b \cdot x^{a}$$

#4: Mass dependence of eigenfrequency
(Resonance 6)

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$f_0 = k^{1/2}/(2\pi) \cdot m^{-1/2}$$
$$y = b \cdot x^{a}$$

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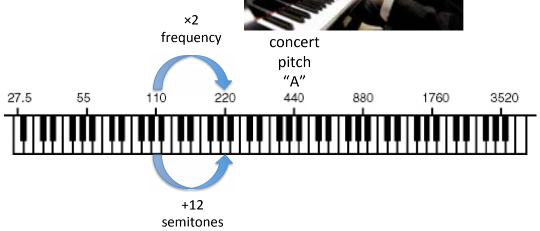
Logarithmic Function: Example



Logarithmic Function

INTEGRAL FORM

$$y = b \cdot \log_a(x)$$

PRACTICAL CONSIDERATIONS:

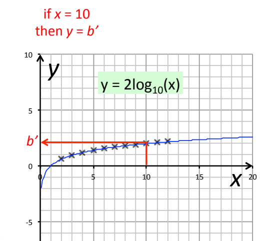
- base is 10 (sometimes *e* or 2)
- if the base is fixed this will modify the factor paramater according to the following identity:

$b \cdot \log_a(x) = b/\log_{10}(a) \cdot \log_{10}(x) = b' \cdot \log_{10}(x)$

VARIABLES: dependent variable, independent variable

$$y = b' \cdot \log_{10}(x)$$

PARAMETERS: factor parameter



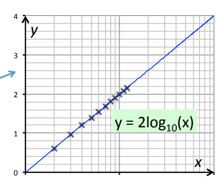
„DIFFERENTIAL" FORM

$$\Delta y \sim \Delta x / x$$

The **absolute change** of the dependent variable is proportional to the **relative change** of the independent variable

Logarithmic Function: Linearization

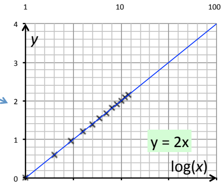
graphical linearization
plot *y* on lin and x on log scales:
*the relationship **looks** linear but it **is** still a log function*



INTEGRAL FORM

$$y = b' \cdot \log_{10}(x)$$

arithmetical linearization
plot *y* as a function of log(*x*):
*the relationship **is** linear*



Logarithmic Function: Some Examples

from the Biophysics Formula Collection ...and elsewhere

#1: The statistical definition of entropy (III.72)

$S = k \ln \Omega$

$$S = k \cdot \log_e(\Omega)$$
$$y = b \cdot \log_a(x)$$

#2: The decibel (dB) scale (VII.10)

$n = 10 \log A_p$

$$n = 10 \cdot \log_{10}(A_p)$$
$$y = b \cdot \log_a(x)$$

#3: The definition of absorbance (VI.34)

$A = \lg(J_0/J)$

$$A = 1 \cdot \log_{10}(J_0/J)$$
$$y = b \cdot \log_a(x)$$

#4: The pH scale

$pH = -\log[H^+]$

$$pH = -1 \cdot \log_{10}([H^+]/(1\ M))$$
$$y = b \cdot \log_a(x)$$

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Summary of Function Types

LINEAR

$\Delta y \propto \Delta x$

The **absolute change** of the dependent variable is proportional to the **absolute change** of the independent variable

y* vs. *x

EXPONENTIAL

$\Delta y / y \propto \Delta x$

The **relative change** of the dependent variable is proportional to the **absolute change** of the independent variable

log*y* vs. *x*

Linearization

y* vs. log*x

LOGARITHMIC

$\Delta y \propto \Delta x / x$

The **absolute change** of the dependent variable is proportional to the **relative change** of the independent variable

log*y* vs. log*x*

POWER

$\Delta y / y \propto \Delta x / x$

The **relative change** of the dependent variable is proportional to the **relative change** of the independent variable

Derivative and Integral: Example #1

Δ Δ

x	$y = x^2$	$y' = \Delta y/\Delta x$	$y'' = \Delta(\Delta y/\Delta x)/\Delta x$
0	0		
1	1	1	
2	4	3	2
3	9	5	2
4	16	7	2
5	25	9	2
6	36	11	2
7	49	13	2
8	64	15	2
9	81	17	2
10	100	19	2

Σ Σ

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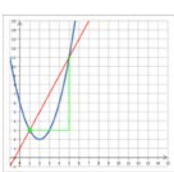
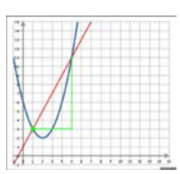
Derivative: Slope of Tangent Line

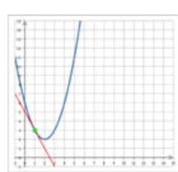
difference quotient:
$\Delta y/\Delta x$
slope of **secant** line

$\Delta \longrightarrow \mathrm{d}$

derivative:
dy/dx
slope of **tangent** line







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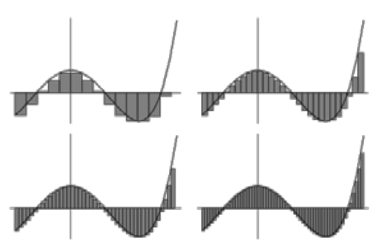
Integral: Area Under the Curve (AUC)

$\Sigma \longrightarrow \int$



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Rectilinear Motions

Quantities, Units, and Equations

displacement: $\Delta s = s_2 - s_1$ — $[\Delta s]$ = m
velocity: $v = \Delta s/\Delta t$ — $[v]$ = m/s
acceleration: $a = \Delta v/\Delta t$ — $[a]$ = m/s^2

Uniform rectilinear motion
$s_t = s_0 + v\cdot t$
v = parameter
$a = 0$

Uniform rectilinear acceleration
$s_t = s_0 + v_0\cdot t + a/2\cdot t^2$
$v_t = v_0 + a\cdot t$
a = parameter

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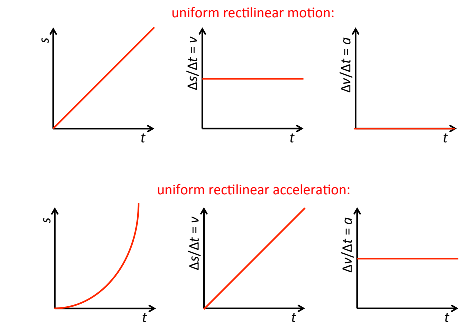
Derivative and Integral: Application

Rectilinear Motions



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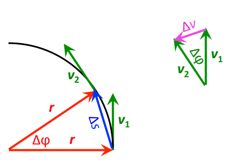
Circular Motion

Quantities, Units, and Equation

angular displacement: $\Delta\varphi = \varphi_2 - \varphi_1$ — $[\Delta\varphi]$ = rad
angular velocity, angular frequency: $\omega = \Delta\varphi/\Delta t$ — $[\omega]$ = rad/s
tangential velocity: $v = \Delta s/\Delta t = r\cdot\Delta\varphi/\Delta t = r\cdot\omega$ — $[v]$ = m/s

centripetal acceleration: $a_{cp} = v^2/r = r\cdot\omega^2$ — $[a]$ = m/s^2



(1) approximation in case of small angles:
displacement = arc length = $v\cdot\Delta t \approx \Delta s$

(2) due to similarity:
$\Delta v/v = \Delta s/r$

(1) + (2):
$\Delta v/v = v\cdot\Delta t/r$

$a_{cp} = v^2/r$

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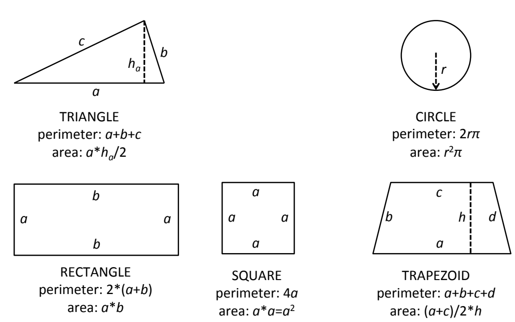
Perimeter & Area



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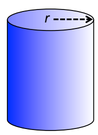
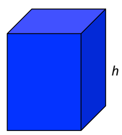
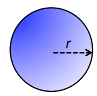
Surface & Volume



CYLINDER (open)
surface (wall only): $2r\pi * h$
volume: $r^2\pi * h$



PRISM (open)
surface (wall only):
(perimeter of base)*h
volume: (area of base)*h



SPHERE
surface: $4r^2\pi$
volume: $4r^3\pi/3$

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Units – SI Base & Derived Units

The SI base units

physical quantity	symbol	unit	symbol
length	*l, x, s, d*	meter	m
mass	*m*	kilogram	kg
time	*t*	second	s
temperature	*T*	kelvin	K
electric current	*I*	ampere	A
amount of substance	*n, N*, ν [nu]	mole	mol
luminous intensity	I_v	candela	cd

Some SI derived units

physical quantity	symbol	unit	symbol	derivation
speed	*v, c*	–	–	$m \cdot s^{-1}$
acceleration	*a*	–	–	$m \cdot s^{-2}$
force	*F*	newton	N	$kg \cdot m \cdot s^{-2}$
energy	*E*	joule	J	$kg \cdot m^2 \cdot s^{-2}$
power	*P*	watt	W	$kg \cdot m^2 \cdot s^{-3}$
intensity	*I*	–	–	$kg \cdot s^{-3}$
pressure	*p*	pascal	Pa	$kg \cdot m^{-1} \cdot s^{-2}$

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Units – SI Prefixes

prefix	symbol	meaning	etymology
exa	E	$\times 10^{18} = \times 1000^6$	Greek 6 (ἕξ = hex)
peta	P	$\times 10^{15} = \times 1000^5$	Greek 5 (πέντε = pente)
tera	T	$\times 10^{12} = \times 1000^4$	Greek 4 (τέτταρες = tettares), originally: monster (τέρας = teras)
giga	G	$\times 10^{9} = \times 1000^3$	Greek giant (γίγας = gigas)
mega	M	$\times 10^{6} = \times 1000^2$	Greek great (μέγας = megas)
kilo	k	$\times 10^{3} = \times 1000^1$	Greek 1000 (χίλιοι = khilioi)
hekto	h	$\times 10^{2}$	Greek 100 (ἑκατόν = hekaton)
deca	da (dk)	$\times 10^{1}$	Greek 10 (δέκα = deka)
deci	d	$\times 10^{-1}$	Latin 10 (decem)
centi	c	$\times 10^{-2}$	Latin 100 (centum)
milli	m	$\times 10^{-3} = \times 1000^{-1}$	Latin 1000 (mille, *pl.* milia)
micro	μ	$\times 10^{-6} = \times 1000^{-2}$	Greek small (μικρός = mikros)
nano	n	$\times 10^{-9} = \times 1000^{-3}$	Greek dwarf (νᾶνος = nanos)
pico	p	$\times 10^{-12} = \times 1000^{-4}$	Spanish small, bit (pico)
femto	f	$\times 10^{-15} = \times 1000^{-5}$	Danish 15 (femten)
atto	a	$\times 10^{-18} = \times 1000^{-6}$	Danish 18 (atten)

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Units – Conversion

Practice Problems: see Excel Sheet on the Website

from "with prefix" to "no prefix":
15 km = 15 · 10^3 m
15 cg = 15 · 10^{-2} g

from "no prefix" to "with prefix":
15 m = 15 / 10^3 km
15 g = 15 / 10^{-2} cg

from "with prefix" to "with other prefix":
15 km = 15 · 10^3 m = 15 · 10^3 / 10^{-2} cm

when the unit has an exponent:
15 km^3 = 15 · $(10^3\ m)^3$ = 15 · $(10^3)^3$ m^3
15 m^3 = 15 / $(10^3)^3$ km^3

liters to and from cubic meters:
1 m^3 = 10 hL = 1000 L
1 dm^3 = 1 L
1 cm^3 = 1 mL
1 mm^3 = 1 μL

time to seconds:
2 days 3 h 12 min 30 s = ((2·24+3)·60+12)·60+30 s

degrees, minutes of arc, seconds of arc:
45° 40′ 30″ = $(45+40/60+30/60^2)$°

degrees to and from radians:
1 rad = (360/2π)°
1° = (2π/360) rad

compound units:
15 kg/m^3 = 15 · 10^3 / $(1/(10^{-2})^3)$ g/cm^3
45 km/h = 45 · 10^3 / 3600 m/s

degrees Celsius to and from kelvins:
T = 15 °C = (15+273) K temperature point
T = 15 K = (15–273) °C temperature point
ΔT = 15 °C = 15 K temperature difference
ΔT = 15 K = 15 °C temperature difference

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