Evans-Searles FT
Crooks FT
Jarzynski equality

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Evans-Searles FT

Denis J Evans, Ezechiel DG Cohen, Gary P Morriss (1993)
Denis J Evans, Debra J Searles (1994)

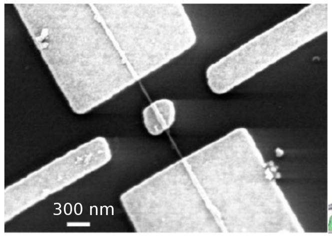
$$\frac{P(\Omega_t = A)}{P(\Omega_t = -A)} = e^{At}$$

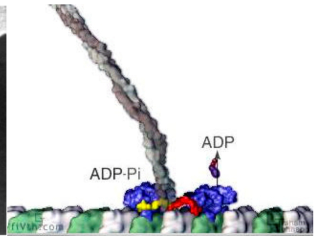
where Ω_t is the average of the entropy production for time t

$$\frac{P(\Omega = S)}{P(\Omega = -S)} = e^{S}$$

Evans and Searles (2002) Advances in Physics, 51: 1529

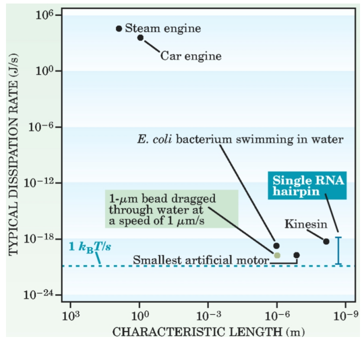
Enzymes and nano size engines





Bustamante et al. (2005) arXiv preprint cond-mat/0511629

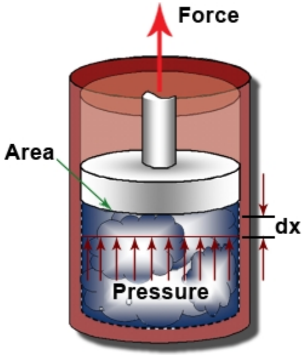
Enzymes and nano size engines



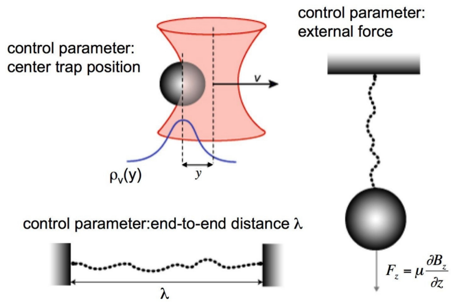
Bustamante et at. (2005) arXiv preprint cond-mat/0511629

Control parameter

For small systems, the control parameter plays the role of the external variables (such as temperature, pressure, volume) used to specify the state of the system in macroscopic thermodynamic systems.



Control parameter



Bustamante, et al. (2005) arXiv preprint cond-mat/0511629.

Crooks FT

For a small driven system which is in contact with a thermostat:

$$\frac{P_F(A \rightarrow B, W)}{P_R(A \leftarrow B, -W)} = e^{\frac{W-\Delta G}{k_B T}}$$

W is the work done when the system is driven from the state A of the control parameter to B.

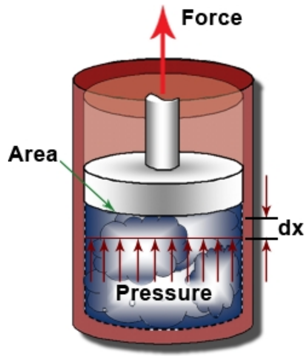
ΔG is the free enthalpy difference between the states A and B

G. E. Crooks, J. Stat. Phys. (1998) 90: 1481

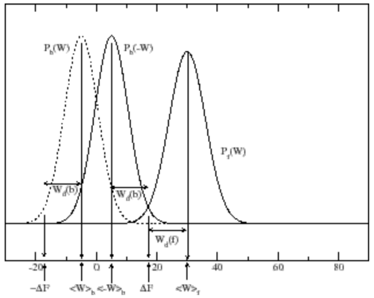
Illustration of the Crooks FT

Both the forward (F) and reverse (R) paths are started from equilibrium.

$$\frac{P_F(A \rightarrow B, W)}{P_R(A \leftarrow B, -W)} = e^{\frac{W-\Delta G}{k_B T}}$$



Crooks FT – distribution of microscopic work



When can we use the Crooks FT

Systems that meet the basic assumptions of molecular dynamics calculations and experiments:

- equilibrium steady state system with time-symmetric microscopic dynamics
- processes that start at equilibrium (it is not necessary to go through equilibrium states or end at equilibrium)

Jarzynski equality

Relates the work done during equilibrium processes with the free enthalpy difference of the initial and end states.

$$\left\langle e^{\frac{-W}{k_B T}} \right\rangle = e^{\frac{-\Delta G}{k_B T}}$$

W is the work that is done when the system is moved from the equilibrium state defined by the control parameter A to the equilibrium state determined by the control parameter B.

The transformation is not required to occur through equilibrium states.

C. Jarzynski, Phys. Rev. Lett. (1997) 78: 2690

Jarzynski equality

It creates a bridge between equilibrium thermodynamics and inequilibrium measurements.

During the transformation, the intensive thermodynamic parameters need not be defined.

An equilibration process is allowed to happen at the final value of an extensive control parameter, since this does not involve work.

Relationship between the Jarzynski equality, the Crooks FT and the Evans-Searles FT

Crooks FT can be derived from Evans-Searles FT if the initial state is steady state or equilibrium.

Crooks FT can be derived from more general conditions than the Evans-Searles FT.

The Jarzynski equation can be derived from the Crooks FT ifboth the initial and final states are equilibrium.

The Crooks FT is generally more robust than the Jarzynski equation and gives a more accurate estimation of the free enthalpy based on the experimental results.

Experimental verification of the fluctuation theorems

General strategy:

small system for a short time, under the influence of small forces

energy / work must be measured with the accuracy of a fraction of $k_B{\cdot}T$

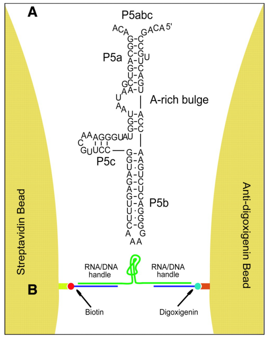
both equilibrium and non-equilibrium ranges should be accessible in the experiments

the experiment must be repeated many times

Verification of the Jarzynski equality

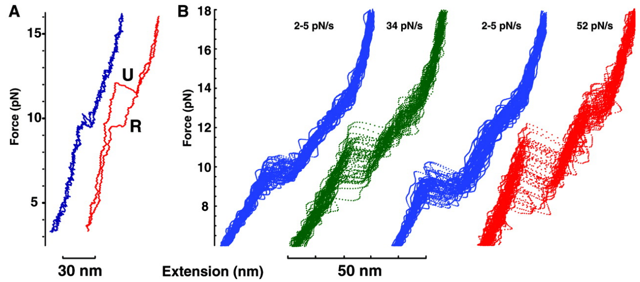
The work done along each trajectory was calculated from the force-displacement function measured using the optical trap.

$$w = \sum F_i \cdot \Delta x_i$$



Liphardt J et al. (2002) Science 296: 1832

Verification of the Jarzynski equality



Liphardt J et al. (2002) Science 296: 1832

Verification of the Jarzynski equality

Three different ways to estimate the free enthalpy difference :

average work
(thermodynamics, quasi-static)

$$W_A = \langle w \rangle$$

based on the fluctuation dissipation theorem (near equilibrium)

$$W_{FD} = \langle w \rangle - \frac{\sigma^2}{2 \cdot k_B T}$$

based on the Jarzynski equality (can be arbitrarily far from equilibrium)

$$W_{JE} = -k_B T \cdot ln \left(\left\langle e^{\frac{-w}{k_B T}} \right\rangle \right)$$

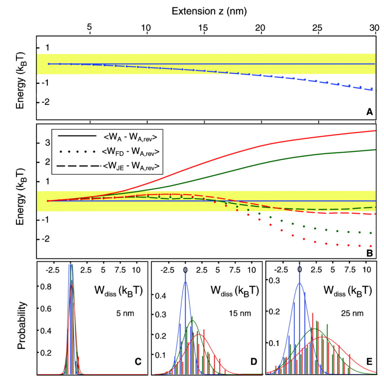
Liphardt J et al. (2002) Science 296: 1832

Verification of the Jarzynski equality

A:
in the reversible range

B:
in the irreversible range

C, D, E:
the distribution of the work w under different conditions



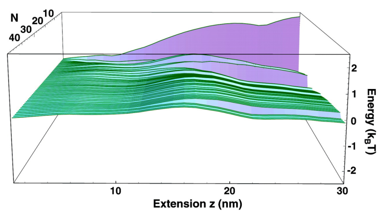
blue: 2-5 pN/s; green: 34 pN/s; red: 52 pN/s

Verification of the Jarzynski equality

The free enthalpy calculated from Jarzynski's equation converges slowly as the number of measurements increases.

$$\Delta G = 59.6 \pm 0.2\ k_B T$$

green: 34 pN/s
red: 52 pN/s



Verification of the Jarzynski equality - summary

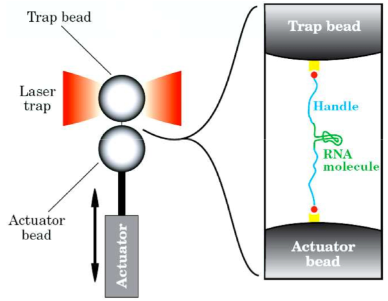
- free enthalpy can be determined with an accuracy of: $0.5\ k_B T$
- the Jarzynski equality gave the best estimate for non-equilibrium measurements (within $1\ k_B T$)
- the Jarzynski equation makes it possible to derive the equilibrium free enthalpy from non-equilibrium measurements
- the free enthalpy calculated from the Jarzynski equation converges slowly if the measurements are done very far from equilibrium (requires many measurements)

Liphardt J et al. (2002) Science 296: 1832

Verification of the Crooks FT

$$\frac{\mathrm{P}(A \to B, W)}{\mathrm{P}(A \leftarrow B, -W)} = e^{\frac{W-\Delta G}{k_B T}}$$

$$W = \sum F_i \cdot \Delta x_i$$



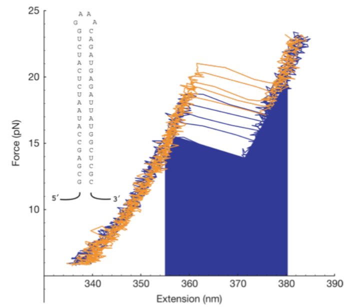
force-guided unfolding of viral RNA using laser tweezers

Collin D et al. (2005) Nature 437: 231

Verification of the Crooks FT

$$W = \sum F_i \cdot \Delta x_i$$

The work done is the integral of the force-elongation curve.



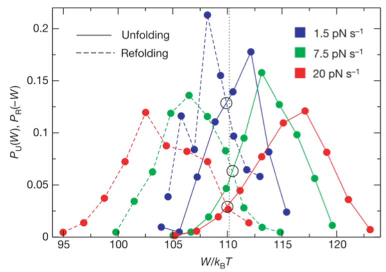
Collin D et al. (2005) Nature 437: 231

Verification of the Crooks FT

Force-controlled unfolding of an RNA hairpin molecule using laser tweezers at different pulling speeds

$\Delta G = 110.3 \pm 0.5\ k_B T$

$$\frac{\mathrm{P}(A \to B, W)}{\mathrm{P}(A \leftarrow B, -W)} = e^{\frac{W-\Delta G}{k_B T}}$$



Collin D et al. (2005) Nature 437: 231

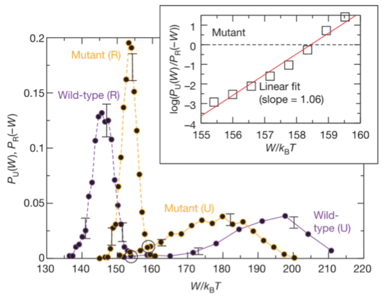
Verification of the Crooks FT

S15 three-helix junction

very far from balance

The probabilities depend on the drawing speed, but their ratio and the location of their intersection do not depend on it.

$$\frac{\mathrm{P}(A \to B, W)}{\mathrm{P}(A \leftarrow B, -W)} = e^{\frac{W-\Delta G}{k_B T}}$$

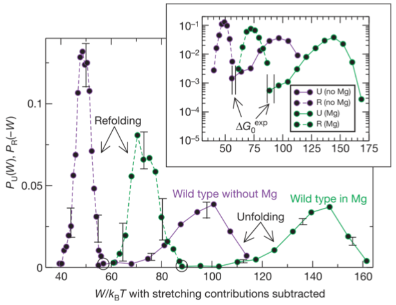


Collin D et al. (2005) Nature 437: 231

The stabilizing effect of Mg $^{2+}$ on RNA estimated based on the Crooks FT

The stabilizing effect of Mg^{2+} on the RNS structure:

$\Delta\Delta G = 31.7 \pm 2\, k_B T$



Collin D et al. (2005) Nature 437: 231

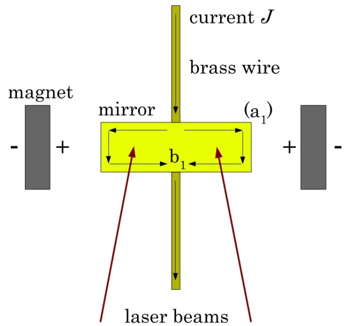
Verification of the Crooks FT - summary

- the Crooks FT described the measurements well (even very far from equilibrium!!)
- equilibrium free enthalpy obtained from non-equilibrium measurements
- the accuracy of the free enthalpy difference: 0.5 $k_B T$
- the RNA structure stabilizing effect of Mg^{2+} ions was measurable

Collin D et al. (2005) Nature 437: 231

Verification of the Jarzynski equality and Crooks FT on a macroscopic system

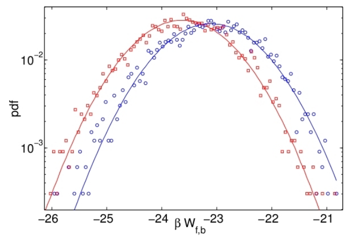
magnetic field controlled deflection of a torsion pendulum



Douarche et al. (2005) Europhysics Letters 70: 593

Verification of the Jarzynski equality and Crooks FT on a macroscopic system

The equations of Crooks FT and Jarzynski correctly describe the Gaussian fluctuations of the isothermal system under investigation.

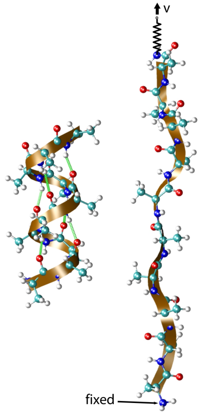


Douarche et al. (2005) Europhysics Letters 70: 593

Using Jarzynski equality in molecular dynamic simulations

unfolding of helical deca-alanine

$\Delta G_{calculated}$ = 21.4 kcal/mol



Park, et al. (2003)
J. Chem. Phys. 119: 3559.

Free enthalpy surface of the mechanical unfolding

$$e^{-\beta A(z)} = \langle e^{-\beta W(z)} \rangle$$

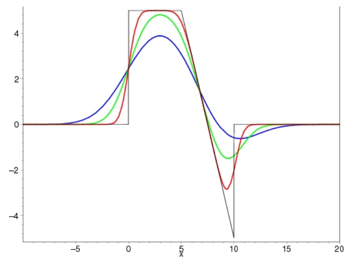
$$W(z) \equiv W[z = z(t)] = \int_{z(0)}^{z(t)} F dz$$

$$e^{-\beta A(z)} = \int dq e^{-\beta G_0(q) - \beta k (q-z)^2/2}$$

Hummer and Szabó (2010) PNAS 107: 21441

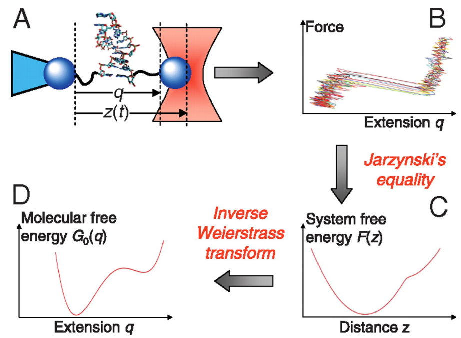
Weierstrass transformation!

$$F(x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} f(y)\, e^{-\frac{(x-y)^2}{4}}\, dy$$



$$e^{-\beta A(z)} = \int dq e^{-\beta G_0(q) - \beta k (q-z)^2/2}$$

Free enthalpy surface of the mechanical unfolding



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