Mathematical and Physical Basis of Medical Biophysics

Lecture 2

Mathematics Necessary for Understanding Physics. Physical Quantities and Units. Kinematics

10th September 2021

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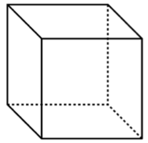
Power Function: Example

mass ∝ volume ∝ [body]length^3
surface area ∝ [body]length^2



Power Function

INTEGRAL FORM

VARIABLES: dependent variable, independent variable

$y = b \cdot x^a$

PARAMETERS: pre-exponential coefficient (b), exponent (a)

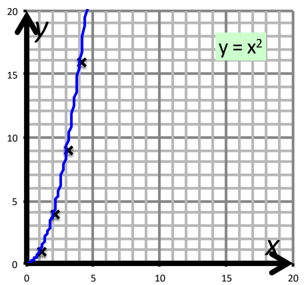
explicit for y: $y = b \cdot x^a$
explicit for x: $x = (y / b)^{1/a}$

"DIFFERENTIAL" FORM

$$\Delta y/y \propto \Delta x/x$$

The **relative change** of the dependent variable is proportional to the **relative change** of the independent variable

if $x = 1$ then $y = b$



inverse proportionality and **square root** functions are also power functions

$$y = \frac{b}{x} = b \cdot x^{-1}$$

$$y = \sqrt{x} = x^{\frac{1}{2}}$$

Power Function: Linearization

graphical linearization
plot both *y* and x on log scales:
*the relationship **looks** linear but it **is** still power function*

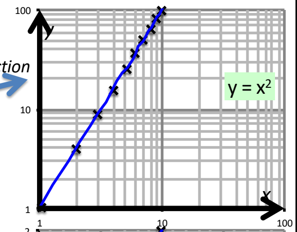
INTEGRAL FORM

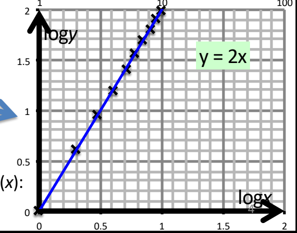
$y = b \cdot x^a$

$$\log y = \log\left(b \cdot x^a\right)$$
$$\log y = \log b + \log\left(x^a\right)$$
$$\log y = \log b + a \cdot \log x$$
$$\underbrace{\log y}_{y} = a \cdot \underbrace{\log x}_{x} + \underbrace{\log b}_{b}$$

intercept = log*b*
log1 = 0
slope = *a*
a = 2

arithmetical linearization
plot log(*y*) as a function of log(*x*):
*the relationship **is** linear*



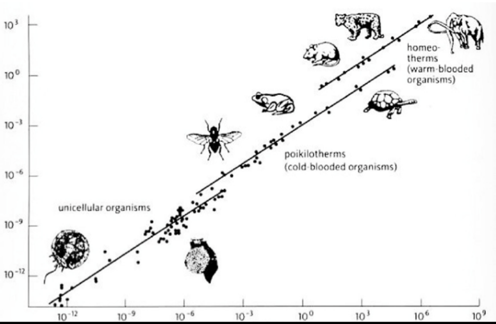


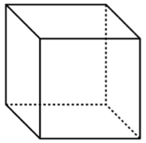
Power Function: Example

Allometric scaling
(E.g. Kleiber's law)

mass ∝ volume ∝ [body]length^3
surface area ∝ [body]length^2

hourly heat production ∝ body mass$^{3/4}$





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Power Function: Some Examples

from the Biophysics Formula Collection

#1: The de Broglie wavelength
(I.3)

$\lambda = h/p$

$\lambda = h \cdot p^{-1}$

$y = b \cdot x^{a}$

#2: Stefan–Boltzmann law
(II.41)

$M_{black} = \sigma \cdot T^{4}$

$y = b \cdot x^{a}$

#3: Duane–Hunt law
(II.80)

$$\lambda_{min} = \frac{hc}{eU_{anode}}$$

$\lambda_{min} = hc/e \cdot U^{-1}$

$y = b \cdot x^{a}$

#4: Mass dependence of
eigenfrequency
(Resonance 6)

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$f_0 = k^{1/2}/(2\pi) \cdot m^{-1/2}$

$y = b \cdot x^{a}$

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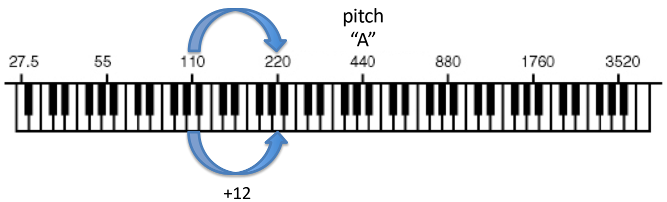
Logarithmic Function: Example

concert
pitch
"A"



Logarithmic Function

INTEGRAL FORM

$y = b \cdot \log_a(x)$

PRACTICAL CONSIDERATIONS:
- base is 10 (sometimes *e* or 2)
- if the base is fixed this will modify the factor paramater according to the following identity:

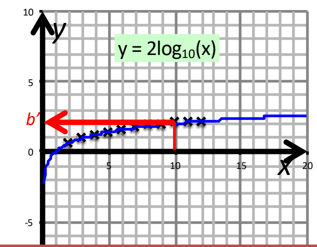
$b \cdot \log_a(x) = b/\log_{10}(a) \cdot \log_{10}(x) = b' \cdot \log_{10}(x)$

VARIABLES: dependent variable, independent variable

$y = b' \cdot \log_{10}(x)$

PARAMETERS: factor parameter

if $x = 10$
then $y = b'$



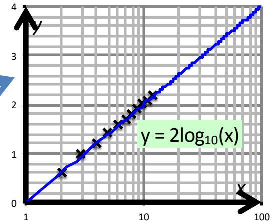
„DIFFERENTIAL" FORM

$\Delta y \sim \Delta x / x$

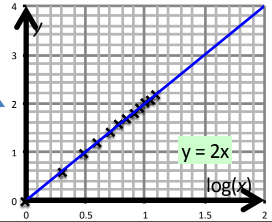
The **absolute change** of the dependent variable is proportional to the **relative change** of the independent variable

Logarithmic Function: Linearization

graphical linearization
plot y on lin and x on log scales:
*the relationship **looks** linear but it **is** still a log function*



INTEGRAL FORM
$y = b' \cdot \log_{10}(x)$



arithmetical linearization
plot y as a function of log(x):
*the relationship **is** linear*

Logarithmic Function: Some Examples

from the Biophysics Formula Collection ...and elsewhere

#1: The statistical definition of entropy
(III.72)
$S = k \ln \Omega$
$S = k \cdot \log_e(\Omega)$
$y = b \cdot \log_a(x)$

#2: The decibel (dB) scale
(VII.10)
$n = 10 \log A_p$
$n = 10 \cdot \log_{10}(A_p)$
$y = b \cdot \log_a(x)$

#3: The definition of absorbance
(VI.34)
$A = \lg(J_0/J)$
$A = 1 \cdot \log_{10}(J_0/J)$
$y = b \cdot \log_a(x)$

#4: The pH scale
$\mathrm{pH} = -\log[\mathrm{H}^+]$
$\mathrm{pH} = -1 \cdot \log_{10}([\mathrm{H}^+]/(1\ \mathrm{M}))$
$y = b \cdot \log_a(x)$

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Functions

Summary

LINEAR FUNCTION
$\Delta y \sim \Delta x$
The **change** of the dependent variable is proportional to the **change** of the independent variable
y vs. x

EXPONENTIAL FUNCTION
$\Delta y/y \sim \Delta x$
The **relative change** of the dependent variable is proportional to the **change** of the independent variable
log$y$ vs. x

Linearization

y vs. log$x$
LOGARITHMIC FUNCTION
$\Delta y \sim \Delta x/x$
The **absolute change** of the dependent variable is proportional to the **relative change** of the independent variable

log$y$ vs. log$x$
POWER FUNCTION
$\Delta y/y \sim \Delta x/x$
The **relative change** of the dependent variable is proportional to the **relative change** of the independent variable

Derivative and Integral: Example #1

Δ Δ

x	y = x²	y' = Δy/Δx	y'' = Δ(Δy/Δx)/Δx
0	0		
1	1		
2	4		
3	9		
4	16		
5	25		
6	36		
7	49		
8	64		
9	81		
10	100		

Σ Σ

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Derivative and Integral: Example #1

Δ Δ

x	y = x^2	y' = Δy/Δx	y'' = Δ(Δy/Δx)/Δx
0	0		
1	1	1	
2	4	3	2
3	9	5	2
4	16	7	2
5	25	9	2
6	36	11	2
7	49	13	2
8	64	15	2
9	81	17	2
10	100	19	2

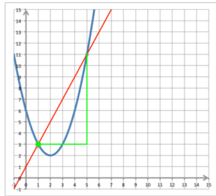
Σ Σ

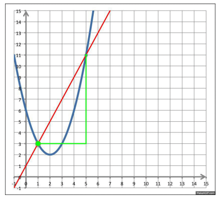
Derivative: slope of tangent line

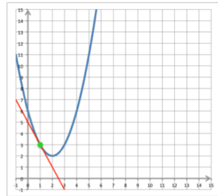
difference quotient:
$\Delta y/\Delta x$
slope of **secant** line

$$\Delta \longrightarrow d$$

derivative:
dy/dx
slope of **tangent** line

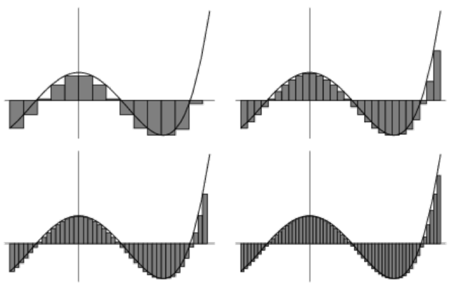






Integral: Area Under the Curve (AUC)

$$\Sigma \longrightarrow \int$$



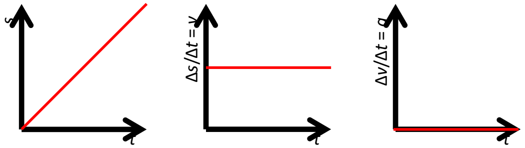
Derivative and Integral: Application

Rectilinear Motion

Derivative and Integral: Application

Rectilinear Motion

uniform rectilinear motion:



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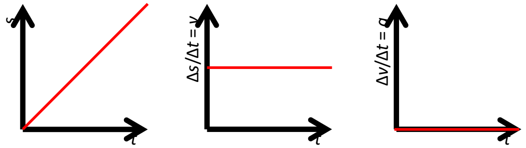
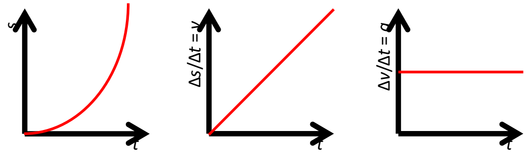
Derivative and Integral: Application

Rectilinear Motion

uniform rectilinear motion:



uniform rectilinear acceleration:



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Circular Motion

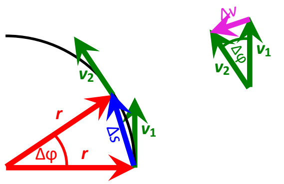
Quantities, Units, and Equation

angular displacement: $\Delta\varphi = \varphi_2 - \varphi_1$ — $[\Delta\varphi]$ = rad

angular velocity, angular frequency: $\omega = \Delta\varphi/\Delta t$ — $[\omega]$ = rad/s

tangential velocity: $v = r \cdot \Delta\varphi/\Delta t = r \cdot \omega$ — $[v]$ = m/s

centripetal acceleration: $a_{cp} = v^2/r = r \cdot \omega^2$ — $[a]$ = m/s²



(1) approximation in case of small angles:
displacement = arc length = $v \cdot \Delta t \approx \Delta s$

(2) due to similarity:
$\Delta v/v = \Delta s/r$

(1) + (2):
$\Delta v/v = v \cdot \Delta t/r$

$a_{cp} = v^2/r$

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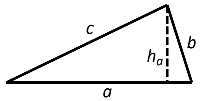
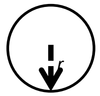
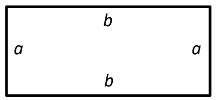
Perimeter & Area



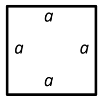
TRIANGLE
perimeter: $a+b+c$
area: $a*h_a/2$



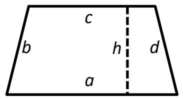
CIRCLE
perimeter: $2r\pi$
area: $r^2\pi$



RECTANGLE
perimeter: $2*(a+b)$
area: $a*b$



SQUARE
perimeter: $4a$
area: $a*a=a^2$



TRAPEZOID
perimeter: $a+b+c+d$
area: $(a+c)/2*h$

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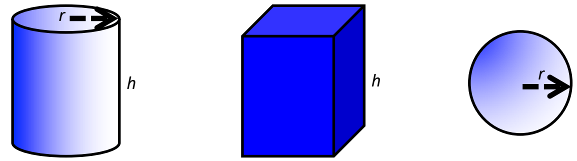
Surface & Volume



CYLINDER (open)	PRISM (open)	SPHERE
surface (wall only): $2r\pi * h$	surface (wall only): (perimeter of base)*h	surface: $4r^2\pi$
volume: $r^2\pi * h$	volume: (area of base)*h	volume: $4r^3\pi/3$

Units – SI Base & Derived Units

The SI base units

physical quantity	symbol	unit	symbol
length	*l, x, s, d*	meter	m
mass	*m*	kilogram	kg
time	*t*	second	s
temperature	*T*	kelvin	K
electric current	*I*	ampere	A
amount of substance	*n, N*, ν [nu]	mole	mol
luminous intensity	I_v	candela	cd

Some SI derived units

physical quantity	symbol	unit	symbol	derivation
speed	*v, c*	–	–	$m \cdot s^{-1}$
acceleration	*a*	–	–	$m \cdot s^{-2}$
force	*F*	newton	N	$kg \cdot m \cdot s^{-2}$
energy	*E*	joule	J	$kg \cdot m^2 \cdot s^{-2}$
power	*P*	watt	W	$kg \cdot m^2 \cdot s^{-3}$
intensity	*I*	–	–	$kg \cdot s^{-3}$
pressure	*p*	pascal	Pa	$kg \cdot m^{-1} \cdot s^{-2}$

Units – SI Prefixes

prefix	symbol	meaning	etymology
exa	E	$\times 10^{18} = \times 1000^6$	Greek 6 (ἕξ = hex)
peta	P	$\times 10^{15} = \times 1000^5$	Greek 5 (πέντε = pente)
tera	T	$\times 10^{12} = \times 1000^4$	Greek 4 (τέτταρες = tettares), originally: monster (τέρας = teras)
giga	G	$\times 10^{9} = \times 1000^3$	Greek giant (γίγας = gigas)
mega	M	$\times 10^{6} = \times 1000^2$	Greek great (μέγας = megas)
kilo	k	$\times 10^{3} = \times 1000^1$	Greek 1000 (χίλιοι = khilioi)
hekto	h	$\times 10^{2}$	Greek 100 (ἑκατόν = hekaton)
deca	da (dk)	$\times 10^{1}$	Greek 10 (δέκα = deka)
deci	d	$\times 10^{-1}$	Latin 10 (decem)
centi	c	$\times 10^{-2}$	Latin 100 (centum)
milli	m	$\times 10^{-3} = \times 1000^{-1}$	Latin 1000 (mille, *pl.* milia)
micro	μ	$\times 10^{-6} = \times 1000^{-2}$	Greek small (μικρός = mikros)
nano	n	$\times 10^{-9} = \times 1000^{-3}$	Greek dwarf (νᾶνος = nanos)
pico	p	$\times 10^{-12} = \times 1000^{-4}$	Spanish small, bit (pico)
femto	f	$\times 10^{-15} = \times 1000^{-5}$	Danish 15 (femten)
atto	a	$\times 10^{-18} = \times 1000^{-6}$	Danish 18 (atten)

Units – Conversion

from "with prefix" to "no prefix":
15 km = 15 · 10^3 m
15 cg = 15 · 10^{-2} g

from "no prefix" to "with prefix":
15 m = 15 / 10^3 km
15 g = 15 / 10^{-2} cg

from "with prefix" to "with prefix":
15 km = 15 · 10^3 m = 15 · 10^3 / 10^{-2} cm

when the unit has an exponent:
15 km³ = 15 · $(10^3\ \text{m})^3$ = 15 · $(10^3)^3$ m³
15 m³ = 15 / $(10^3)^3$ km³

liters to and from cubic meters:
1 m³ = 10 hL = 1000 L
1 dm³ = 1 L
1 cm³ = 1 mL
1 mm³ = 1 μL

time to seconds:
2 days 3 h 12 min 30 s = ((2·24+3)·60+12)·60+30 s

degrees, minutes of arc, seconds of arc:
45° 40′ 30″ = $(45+40/60+30/60^2)°$

degrees to and from radians:
1 rad = (360/2π)°
1° = (2π/360) rad

compound units:
15 kg/m³ = 15 · 10^3 / $(1/(10^{-2})^3)$ g/cm³
45 km/h = 45 · 10^3 / 3600 m/s

degrees Celsius to and from kelvins:
T = 15 °C = (15+273) K
T = 15 K = (15–273) °C
ΔT = 15 °C = 15 K
ΔT = 15 K = 15 °C